Box Spline Reconstruction on the Face-Centered Cubic Lattice

Minho Kim, Alireza Entezari and Jörg Peters

IEEE Visualization 2008
23 October
Overview

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- Integrated with POV-Ray ray-tracer and source codes are freely available at http://www.cise.ufl.edu/research/SurfLab/08vis.
Example

original

standard method

our approach

100%

6%

Cartesian lattice
tri-quadratic B-spline

6%

FCC lattice
6-direction box spline
Example

original
standard method
our approach

100%
6%
6%

Cartesian lattice
tri-quadratic B-spline
FCC lattice
6-direction box spline

For (random) evaluation, our approach is **20%** faster!
Sampling Lattice: FCC Lattice

- Reconstruction
- Lattice
- Filter

FCC lattice
octet-truss
6-direction box spline
FCC Lattice: Definition
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Cubic (Cartesian) lattice

\( \times \sqrt[3]{4} \)
FCC Lattice: Definition

Cubic (Cartesian) lattice $\times \sqrt[3]{4}$ + additional facet points
FCC Lattice: Definition

Cubic (Cartesian) lattice + additional facet points → “Face-Centered Cubic” lattice.
FCC Lattice: Voronoi Cell
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12 nearest neighbor points
FCC Lattice: Voronoi Cell

12 nearest neighbor points
→ Voronoi cell = Rhombic Dodecahedron.
FCC Lattice: Applications
FCC Lattice: Applications

- Sampling efficiency: Cartesian $<$ FCC $<$ BCC. (Petersen & Middleton ’62)
FCC Lattice: Applications

- **Sampling efficiency:** Cartesian $<$ FCC $<$ BCC. (Petersen & Middleton ’62)

  **Efficient sampling:** minimizes number of samples necessary to reconstruct an isotropic band-limited signal.

- Multiresolution data structure (Inoue et al. 2008), Global illumination (Qiu et al. 2007),
Reconstruction Filter: 6-Direction Box Spline

- Reconstruction
- Lattice
  - FCC lattice
- Filter
  - Octet-truss
  - 6-direction box spline
Box-Splines: A Bivariate Example
Box-Splines: A Bivariate Example

Direction matrix
\[
\begin{bmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}
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Box-Splines: A Bivariate Example

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Finite support: Minkowski sum of the directions.
Box-Splines: A Bivariate Example

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- **Finite support**: Minkowski sum of the directions.
- **Piecewise polynomial** of degree
  \((\# \text{ of directions} - \dim \text{ran} \Xi)\).
Box-Splines: A Bivariate Example

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- **Finite support**: Minkowski sum of the directions.
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- Polynomial pieces delineated by the shifts of the knot planes
  (Hyperplanes spanned by the directions of \(\Xi\)).
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- Polynomial pieces join **smoothly**: \(C^{(m(\Xi) - 1)}\).
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◮ **Finite support**: Minkowski sum of the directions.

◮ **Piecewise polynomial** of degree

(# of directions - \text{dim ran } \Xi).

◮ Polynomial pieces delineated by the shifts of the knot planes

(Hyperplanes spanned by the directions of \Xi).

◮ Polynomial pieces join **smoothly**: \( C^{m(\Xi)-1} \).

◮ “Box Splines” (Carl de Boor et al., 1993).
Box Splines vs. B-splines
Box Splines vs. B-splines

In general, compared to tensor-product B-splines with the same polynomial degree, box splines have
Box Splines vs. B-splines

In general, compared to tensor-product B-splines with the same polynomial degree, box splines have

- higher approximation order,
Box Splines vs. B-splines

In general, compared to tensor-product B-splines with the same polynomial degree, box splines have

- higher approximation order,
- smaller support and
Box Splines vs. B-splines

In general, compared to tensor-product B-splines with the same polynomial degree, box splines have

- higher approximation order,
- smaller support and
- higher symmetry.
6-Direction Box Spline
6-Direction Box Spline

Direction matrix

\[
\begin{bmatrix}
1 & -1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 1 & 1 \\
\end{bmatrix}
\]
6-Direction Box Spline

Direction matrix

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\begin{bmatrix}
1 & -1 & 1 & 1 & 0 & 0 \\
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0 & 0 & 1 & -1 & 1 & 1
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Support = **Truncated Octahedron**.
6-Direction Box Spline

- Direction matrix

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\begin{bmatrix}
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1 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 1 & 1 \\
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- Support = **Truncated Octahedron**.

- Total degree cubic and \( C^1 \) **continuous**.
6-Direction Box Spline

- Direction matrix
  \[
  \begin{bmatrix}
  1 & -1 & 1 & 1 & 0 & 0 & 0 \\
  1 & 1 & 0 & 0 & 1 & -1 & 0 \\
  0 & 0 & 1 & -1 & 1 & 1 & 1 \\
  \end{bmatrix}.
  \]

- Support = \textbf{Truncated Octahedron}.
- Total degree cubic and $C^1$ \textit{continuous}.
- Approximation order is 3.
6-Direction Box Spline

- Direction matrix
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  \end{bmatrix}
  \]

- Support = Truncated Octahedron.

- Total degree cubic and \( C^1 \) continuous.

- Approximation order is 3.

- Exact rational coefficients are pre-computed.
6-Direction Box Spline (cont’d)
Polynomial Structure: Octet-Truss
Octet-Truss Structure
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Octet-Truss Structure
Octet-Truss Structure
6-Direction Box Spline on the FCC Lattice
6-Direction Box Spline on the FCC Lattice

reconstruction

lattice

filter

FCC lattice

octet-truss

6-direction box spline
6-Direction Box Spline on the FCC Lattice

reconstruction

lattice

FCC lattice

octet-truss

6-direction box spline

filter
6-Direction Box Spline on the FCC Lattice

- Reconstruction
- Lattice
  - FCC lattice
  - Octet-truss
- Filter
  - 6-direction box spline
6-Direction Box Spline on the FCC Lattice

reconstruction

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6-Direction Box Spline on the FCC Lattice
Polynomial structure → octet-truss structure.
6-Direction Box Spline on the FCC Lattice

- Polynomial structure $\rightarrow$ octet-truss structure.
- Shifts are linearly independent $\rightarrow$ basis functions.
Comparison: Math
### Comparison: Math

<table>
<thead>
<tr>
<th>lattice filter</th>
<th>Standard</th>
<th>Our approach</th>
</tr>
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<tbody>
<tr>
<td></td>
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<td>6</td>
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- **lattice**: Cartesian, FCC
- **filter**: tri-quadratic, 6-direction box spline
- **approximation order**: 3
- **total degree**: 6
## Comparison: Math

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<td><strong>6</strong></td>
<td><strong>3</strong></td>
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<td><strong>27</strong></td>
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<tr>
<td><strong>sampling efficiency</strong></td>
<td>poor</td>
<td>good</td>
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Comparison: Reconstruction (Carp dataset)

- Original
- Standard method: Cartesian lattice, tri-quadratic B-spline, 6% error
- Our approach: FCC lattice, 6-direction box spline, 6% error
Comparison: Reconstruction (Marschner-Lobb function)
Comparison: Reconstruction (Marschner-Lobb function)

density $0.07^{-3}$

Standard

Our approach
Comparison: Reconstruction (Marschner-Lobb function)

density $0.06^{-3}$

Standard

Our approach
Comparison: Reconstruction (Marschner-Lobb function)

density $0.05^{-3}$

Standard  Our approach
## Comparison: Computation Time

<table>
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<tr>
<th>Dataset</th>
<th>Standard</th>
<th>Our approach</th>
<th>Ratio</th>
</tr>
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<tr>
<td>Marschner-Lobb</td>
<td>135</td>
<td>98</td>
<td>72%</td>
</tr>
<tr>
<td>Carp</td>
<td>515</td>
<td>358</td>
<td>69%</td>
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- Rendering time (in seconds) to generate ray-casted images.
Try yourself!

For more information, please visit

http://www.cise.ufl.edu/research/SurfLab/08vis
Thank you!
Selected References

