Topics in Computer Graphics
Chap 14: Tensor Product Patches

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Bilinear Interpolation

- Let $\mathbf{b}_{0,0}, \mathbf{b}_{0,1}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1} \in \mathbb{E}^3$.
- The set of all points $\mathbf{x} \in \mathbb{E}^3$ of the form

  $$\mathbf{x}(u, v) = \sum_{i=0}^{1} \sum_{j=0}^{1} \mathbf{b}_{i,j} B_i^1(u) B_j^1(v)$$

  is called a *hyperbolic paraboloid* through the four points $\{\mathbf{b}_{i,j}\}$.
- In matrix form:

  $$\mathbf{x}(u, v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} \\ \mathbf{b}_{10} & \mathbf{b}_{11} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}.$$

- $\mathbf{x}(u, v)$ is linear in both u and v
  $\rightarrow$ $\mathbf{x}$ is called the *bilinear interpolant*.

Bilinear Interpolation (cont'd)

- Can be viewed as a map of the unit square $0 \leqslant u, v \leqslant 1$ in the u, v-plane. → *domain*
- A line parallel to one of the axes in the domain corresponds to a curve in the range. (Equivalent to fixing either u or v.)
  → *isoparametric curve*
- Every isoparametric curve of the hyperbolic paraboloid is a straigt line.
  → *ruled surface*
- Evaluation

$$\begin{aligned}
\mathbf{b}_{0,0}^{0,1} &= (1-v)\mathbf{b}_{0,0} + v\mathbf{b}_{0,1} \\
\mathbf{b}_{1,0}^{0,1} &= (1-v)\mathbf{b}_{1,0} + v\mathbf{b}_{1,1} \\
\mathbf{x}(u,v) = \mathbf{b}_{0,0}^{1,1}(u,v) &= (1-u)\mathbf{b}_{0,0}^{0,1} + u\mathbf{b}_{1,0}^{0,1}
\end{aligned}$$

Equivalent to computing the coefficients of the isoparametric line $v = const$ first and then evaluating this

Bilinear Interpolation (cont'd)

- Why "hyperbolic paraboloid"?
  - The (nonparametric) surface $z = xy$ can be considered as the bilinear interpolant to the four points (Why?)

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

  - The intersection of the surface and a plane parallel to the xy-plane is a hyperbola.
  - The intersection of the surface and a plane containing the z-axis is a parabola.

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The Direct de Casteljau Algorithm

- Higher degree surfaces can be obtained by repeated application of bilinear interpolation.
- Given points $\{\mathbf{b}_{i,j}\}_{i,j=0}^{n}$ and parameter values $(u, v) \in \mathbb{R}^2$, set

$$\mathbf{b}_{i,j}^{r,r} = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{i,j}^{r-1,r-1} & \mathbf{b}_{i,j+1}^{r-1,r-1} \\ \mathbf{b}_{i+1,j}^{r-1,r-1} & \mathbf{b}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}, \qquad i, j = 0, \ldots, n-r$$

  and $\mathbf{b}_{i,j}^{0,0} = \mathbf{b}_{i,j}$. Then $\mathbf{b}_{0,0}^{n,n}(u, v)$ is the point with parameter values (u, v) on the *Bézier surface* $\mathbf{b}^{n,n}$.
- $\{\mathbf{b}_{i,j}\}$: Bézier points or control points
- The net of $\{\mathbf{b}_{i,j}\}$: Bézier net or control net of the surface $\mathbf{b}^{n,n}$
- What if the degrees in u and v are different? $(\{\mathbf{b}_{i,j}\}_{i=0,\ldots,m, j=0,\ldots,n})$

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The Tensor Product Approach

- "A surface is the locus of a curve that is moving through space and thereby changing its shape."
- Formulation
  1. Given the initial Bézier curve of degree m:
  $$\mathbf{b}^m(u) = \sum_{i=0}^{m} \mathbf{b}_i B_i^m(u)$$
  2. Let each control point $\mathbf{b}_i$ traverses a Bézier curve of degree n:
  $$\mathbf{b}_i = \mathbf{b}_i(v) = \sum_{j=0}^{n} \mathbf{b}_{i,j} B_j^n(v).$$
  3. Combining these two, we obtain the point $\mathbf{b}^{m,n}(u,v)$ on the surface $\mathbf{b}^{m,n}$ as
  $$\mathbf{b}^{m,n}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{b}_{i,j} B_i^m(u) B_j^n(v).$$

The Tensor Product Approach (cont'd)

- For a fixed $\hat{v} = const$, its Bézier points are obtained by

$$\mathbf{b}_{i,0}^{0,n}(\hat{v}) = \sum_{j=0}^{n} \mathbf{b}_{ij} B_j^n(\hat{v}), \quad i = 0, \ldots, m.$$

- Other straight lines in the domain (i.e. diagonals) are mapped to higher-degree (generally $m + n$) curves on the patch.

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Properties

- Affine invariance

$$\sum_{j=0}^{n} \sum_{i=0}^{m} B_i^m(u) B_j^n(v) \equiv 1(u, v)$$

  $\rightarrow \mathbf{b}^{m,n}(u, v)$ is an affine combination of the control points.
- Convex hull property
- Boundary curves of the patch $\mathbf{b}^{m,n}(u, v)$ are polynomial curves.
- Variation diminishing property $\rightarrow$ NOT inherited

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Degree Elevation

- To rewrite a Bézier surface of degree (m, n) as one of degree $(m+1, n)$:

$$\mathbf{b}^{m,n}(u,v) = \sum_{j=0}^{n} \left[\sum_{i=0}^{m+1} \mathbf{b}_{i,j}^{(1,0)} B_i^{m+1}(u) \right] B_j^n(v).$$

$$\mathbf{b}_{i,j}^{(1,0)} = \frac{i}{m+1}\mathbf{b}_{i-1,j} + \left(1 - \frac{i}{m+1}\right)\mathbf{b}_{i,j}, \quad \begin{cases} i = 0, \ldots, m+1 \\ j = 0, \ldots, n. \end{cases}$$

- as one of degree $(m+1, n+1)$:

$$\mathbf{b}_{i,j}^{(1,1)} = \begin{bmatrix} \frac{i}{m+1} & 1 - \frac{i}{m+1} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{i-1,j-1} & \mathbf{b}_{i-1,j} \\ \mathbf{b}_{i,j-1} & \mathbf{b}_{i,j} \end{bmatrix} \begin{bmatrix} \frac{j}{n+1} \\ 1 - \frac{j}{n+1} \end{bmatrix}$$

$$\begin{cases} i = 0, \ldots, m+1, \\ j = 0, \ldots, n+1. \end{cases}$$

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Derivatives

- Partial derivatives along u and v.
- Tangent vector of an isoparametric curve.

$$\begin{aligned}\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u,v) &= \sum_{j=0}^{n}\left[\frac{\partial}{\partial u}\sum_{i=0}^{m}\mathbf{b}_{i,j}B_i^m(u)\right]B_j^n(v)\\ &= m\sum_{j=0}^{n}\sum_{i=0}^{m}\Delta^{(1,0)}\mathbf{b}_{i,j}B_i^{m-1}(u)B_j^n(v).\end{aligned}$$

where $\Delta^{(1,0)}\mathbf{b}_{i,j} := \mathbf{b}_{i+1,j} - \mathbf{b}_{i,j}$.

$$\frac{\partial}{\partial v}\mathbf{b}^{m,n}(u,v) = n\sum_{i=0}^{m}\sum_{j=0}^{n}\Delta^{(0,1)}\mathbf{b}_{i,j}B_j^{n-1}(u)B_i^m(u).$$

Higher Derivatives

$$\frac{\partial^r}{\partial u^r}\mathbf{b}^{m,n}(u,v) = \frac{m!}{(m-r)!}\sum_{j=0}^{n}\sum_{i=0}^{m-r}\Delta^{(r,0)}\mathbf{b}_{i,j}B_i^{m-r}(u)B_j^n(v)$$

where $\Delta^{(r,0)}\mathbf{b}_{i,j} := \Delta^{(r-1,0)}\mathbf{b}_{i+1,j} - \Delta^{(r-1,0)}\mathbf{b}_{i,j}$ and

$$\frac{\partial^s}{\partial v^s}\mathbf{b}^{m,n}(u,v) = \frac{n!}{(n-s)!}\sum_{i=0}^{m}\sum_{j=0}^{n-s}\Delta^{(0,s)}\mathbf{b}_{i,j}B_j^{n-s}(v)B_i^m(u).$$

where $\Delta^{(0,s)}\mathbf{b}_{i,j} := \Delta^{(0,s-1)}\mathbf{b}_{i,j+1} - \Delta^{(0,s-1)}\mathbf{b}_{i,j}$.

Mixed Partial Derivatives & Cross Boundary Derivatives

- Mixed partial derivatives

$$\frac{\partial^{r+s}}{\partial u^r \partial v^s}\mathbf{b}^{m,n}(u,v) = \frac{m!n!}{(m-r)!(n-s)!}\sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\Delta^{r,s}\mathbf{b}_{i,j}B_i^{m-r}(u)B_j^{n-s}(v).$$

- Cross boundary derivatives

$$\frac{\partial^r}{\partial u^r}\mathbf{b}^{m,n}(0,v) = \frac{m!}{(m-r)!}\sum_{j=0}^{n}\Delta^{(r,0)}\mathbf{b}_{0,j}B_j^n(0).$$

$\rightarrow$ Important when we formulate condition for C^r continuity between adjacent patches.

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Blossoms

$$\mathbf{b}[u_1, \ldots, u_m | v_1, \ldots, v_n]$$

1. Compute the (curve) blossom value $\mathbf{b}_i[u_1, \ldots, u_m]$ of all rows of control points, using the same values for each row.
2. Then use those values as input to the (curve) blossom $\mathbf{b}[v_1, \ldots, v_n]$.

- $\mathbf{b}[u^{<m>} | v^{<n>}]$ is the point on the surface.
- The order of evaluation does not matter.
- Multiaffinity in both u and v.

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Curves on a Surface

1. Let $\mathbf{p} = (\mathbf{p}_u, \mathbf{p}_v)$ and $\mathbf{q} = (\mathbf{q}_u, \mathbf{q}_v)$ be u, v-coordinates of two points $\mathbf{p}$ and $\mathbf{q}$ in the domain of a tensor product patch of degree (n, n).
2. A line through $\mathbf{p}$ and $\mathbf{q}$:

$$\mathbf{u}(t) = (1-t)\mathbf{p} + t\mathbf{q}.$$

3. A point on the curve is given by

$$\mathbf{b}[((1-t)\mathbf{p}_u + t\mathbf{q}_u)^{<n>}|((1-t)\mathbf{q}_v + t\mathbf{q}_v)^{<n>}].$$

4. Applying the Leibniz formula to the u-part,

$$\sum_{i+j=n} \binom{n}{i,j} (1-t)^i t^j \mathbf{b}[\mathbf{p}_u^{<i>}\mathbf{q}_u^{<j>}|((1-t)\mathbf{p}_v + t\mathbf{q}_v)^{<n>}]$$

Curves on a Surface (cont'd)

5. Applying the Leibniz formula to the v-part,

$$\sum_{i+j=n} \sum_{r+s=n} \binom{n}{i,j} \binom{n}{r,s} (1-t)^i t^j (1-t)^r t^s \mathbf{b}[\mathbf{p}_u^{<i>}, \mathbf{q}_u^{<j>} | \mathbf{p}_v^{<r>}, \mathbf{q}_v^{<s>}]$$

6. Equivalently,

$$\begin{aligned}
&\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n}{j} (1-t)^{2n-i-j} t^{i+j} \mathbf{b}[\mathbf{p}_u^{<n-i>}, \mathbf{q}_u^{<i>} | \mathbf{p}_v^{<n-j>}, \mathbf{q}_v^{<j>}] \\
&= \sum_{k=0}^{2n} \sum_{i+j=k} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{k}} B_k^{2n}(t) \mathbf{b}[\mathbf{p}_u^{<n-i>}, \mathbf{q}_u^{<i>} | \mathbf{p}_v^{<n-j>}, \mathbf{q}_v^{<j>}] \\
&= \sum_{k=0}^{2n} \left(\sum_{i+j=k} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{k}} \mathbf{b}[\mathbf{p}_u^{<n-i>}, \mathbf{q}_u^{<i>} | \mathbf{p}_v^{<n-j>}, \mathbf{q}_v^{<j>}] \right) B_k^{2n}(t) \\
&=: \sum_{k=0}^{2n} \mathbf{c}_k B_k^{2n}(t)
\end{aligned}$$

$\rightarrow$ A curve of degree $2n$

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Normal Vectors

- Required for lighting computation.
- Can be obtained by cross product of any two tangent vectors at the given point on the surface.

$$\mathbf{b}(u, v) = \frac{\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u, v) \wedge \frac{\partial}{\partial v}\mathbf{b}^{m,n}(u, v)}{\|\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u, v) \wedge \frac{\partial}{\partial v}\mathbf{b}^{m,n}(u, v)\|}$$

- Degenerate cases?

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The Matrix Form of a Bézier patch

$$\mathbf{b}^{m,n}(u,v) = \begin{bmatrix} B_0^m(u) & \cdots & B_m^m(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{00} & \cdots & \mathbf{b}_{0n} \\ \vdots & \ddots & \vdots \\ \mathbf{b}_{n0} & \cdots & \mathbf{b}_{nn} \end{bmatrix} \begin{bmatrix} B_0^n(v) \\ \vdots \\ B_n^n(v) \end{bmatrix}$$

- $\{\mathbf{b}_{ij}\}$: Geometry matrix of the patch

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