Topics in Computer Graphics
Chap 8: B-Spline Curves

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Outline

Motivation

- Consists of polynomial pieces (e.g. piecewise linear curve)
- Bézier case (quadratic)
  - Control points: $\mathbf{b}[0,0], \mathbf{b}[0,1], \mathbf{b}[1,1]$
  - Obtained from the sequence $0,0,1,1$ by taking successive pairs: $\{\boxed{0,0},1,1\}$, $\{0,\boxed{0,1},1\}$, $\{0,0,\boxed{1,1}\}$
- Generalizing to a sequence u_0, u_1, u_2, u_3

$$
\begin{aligned}
\mathbf{b}[u,u] &= \frac{u_2-u}{u_2-u_1}\mathbf{b}[u_1,u] + \frac{u-u_1}{u_2-u_1}\mathbf{b}[u,u_2] \\
&= \frac{u_2-u}{u_2-u_1}\left(\frac{u_2-u}{u_2-u_0}\mathbf{b}[u_0,u_1] + \frac{u-u_0}{u_2-u_0}\mathbf{b}[u_1,u_2]\right) \\
&\quad + \frac{u-u_1}{u_2-u_1}\left(\frac{u_3-u}{u_3-u_1}\mathbf{b}[u_1,u_2] + \frac{u-u_1}{u_3-u_1}\mathbf{b}[u_2,u_3]\right)
\end{aligned}
$$

  - Based on the identity $u = \frac{u_i-u}{u_i-u_j}u_j + \frac{u-u_j}{u_i-u_j}u_i$
  - Successively express u in terms of intervals of growing size
  - Figure 8.1

de Boor Algorithm

$$
\begin{array}{lll}
\mathbf{b}[u_0, u_1] & & \\
\mathbf{b}[u_1, u_2] & \mathbf{b}[u_1, u] & \\
\mathbf{b}[u_2, u_3] & \mathbf{b}[u, u_2] & \mathbf{b}[u, u]
\end{array}
$$

- de Boor generalization of de Casteljau algorithm
- Try Example 8.1.

Outline

Definitions of U and U_i^r

- $U := [u_I, u_{I+1}] \subset \{u_i\}$
- Let the ordered set U_i^r defined such that
  - consists of $r+1$ successive knots
  - u_I is the $(r-i)$th element of U_i^r with "0th" denoting the first of U_i^r's elements.
- $U_i^r := \{u_{I-r+i}, \ldots, u_{I+i}\}$
- $U_i^r = U_i^{r+1} \cap U_{i+1}^{r+1} =$ $\{u_{I-r+i-1}, \boxed{u_{I-r+i}, \ldots, u_{I+i}}\} \cap \{\boxed{u_{I-r+i}, \ldots, u_{I+i}}, u_{I+i+1}\}$
- When we refer to the U_i^r as intervals, $U_i^r := [u_{I-r+i}, u_{I+i}]$
- $U_1^1 = [u_I, u_{I+1}] = U$
- $U_0^0 = [u_I, u_I] = u_I$ and $U_1^0 = [u_{I+1}, u_{I+1}] = u_{I+1}$

Curve Segement Corresponding to $U = [u_I, u_{I+1}]$

- A degree n curve segment corresponding to the interval U is given by $n+1$ control points $\mathbf{d}_i$ which are defined by

$$\mathbf{d}_i := \mathbf{b}[U_i^{n-1}] = \mathbf{b}[u_{I-n+i+1}, \ldots, u_{I+i}], \quad i = 0, \ldots, n$$

$$\begin{aligned}
\mathbf{d}_0 \quad &:= \mathbf{b}[u_{I-n+1}, u_{I-n+2}, \ldots, u_{I-1}, \boxed{u_I}] \\
\mathbf{d}_1 \quad &:= \mathbf{b}[u_{I-n+2}, u_{I-n+3}, \ldots, \boxed{u_I, u_{I+1}}] \\
&\vdots \\
\mathbf{d}_{n-1} \quad &:= \mathbf{b}[\boxed{u_I, u_{I+1}} \ldots, u_{I+n-2}, u_{I+n-1}] \\
\mathbf{d}_n \quad &:= \mathbf{b}[\boxed{u_{I+1}}, u_{I+2}, \ldots, u_{I+n-1}, u_{I+n}]
\end{aligned}$$

- A point $\mathbf{x}(u) = \mathbf{b}[u^{<n>}]$ on the curve is recursively computed as

$$\mathbf{d}_i^r(u) = \mathbf{b}[u^{<r>}, U_i^{n-1-r}] \quad r = 1, \ldots, n \text{ and } i = 0, \ldots, n-r$$

with $\mathbf{x}(u) = \mathbf{d}_0^n(u) = \mathbf{b}[u^{<n>}]$.
  - de Boor algorithm

de Boor Algorithm

Using

$$u = \frac{u_{I+i+1} - u}{u_{I+i+1} - u_{I-n+r+i}} u_{I-n+r+i} + \frac{u - u_{I-n+r+i}}{u_{I+i+1} - u_{I-n+r+i}} u_{I+i+1}$$

$$\begin{aligned}
\mathbf{d}_i^r(u) &= \mathbf{b}[u^{<r>}, U_i^{n-1-r}] = \mathbf{b}[u^{<r-1>}, \boxed{u}, u_{I-n+r+i+1}, \ldots, u_{I+i}] \\
&= \frac{u_{I+i+1} - u}{u_{I+i+1} - u_{I-n+r+i}} \mathbf{b}[u^{<r-1>}, \boxed{u_{I-n+r+i}}, u_{I-n+r+i+1}, \ldots, u_{I+i}] \\
&+ \frac{u - u_{I-n+r+i}}{u_{I+i+1} - u_{I-n+r+i}} \mathbf{b}[u^{<r-1>}, u_{I-n+r+i+1}, \ldots, u_{I+i}, \boxed{u_{I+i+1}}] \\
&= (1 - t_{i+1}^{n-r+1}) \mathbf{b}[u^{<r-1>}, U_i^{(n-1)-(r-1)}] + t_{i+1}^{n-r+1} \mathbf{b}[u^{<r-1>}, U_{i+1}^{(n-1)-(r-1)}] \\
&= (1 - t_{i+1}^{n-r+1}) \mathbf{d}_i^{r-1}(u) + t_{i+1}^{n-r+1} \mathbf{d}_{i+1}^{r-1}(u) \quad r = 1, \ldots, n \text{ and } i = 0, \ldots, n-r
\end{aligned}$$

where $t_{i+1}^{n-r+1} := \frac{u - u_{I-n+r+i}}{u_{I+i+1} - u_{I-n+r+i}}$ is the local parameter in the interval $U_{i+1}^{n-r+1} = [u_{I-n+r+i}, u_{I+i+1}]$.

- Figure 8.2

de Boor Algorithm vs. de Casteljau Algorithm

With the knot sequence $\{0^{<n>}, 1^{<n>}\}$ and $U = [0, 1]$,

$\mathbf{d}_i^r(u) = \mathbf{b}[u^{<r>}, 0^{<n-r-i>}, 1^{<i>}], \quad r = 1, \ldots, n$ and $i = 0, \ldots, n-r$.

$\rightarrow$ de Casteljau algorithm (4.10 on p.52)

Derivatives of a B-Spline Curve Segment

- First derivative:

$$\dot{\mathbf{x}}(u) = n\mathbf{b}[u^{<n-1>}, \vec{1}] = \frac{n}{|U|}(\mathbf{d}_1^{n-1} - \mathbf{d}_0^{n-1})$$

  where $|U| = U_1^0 - U_0^0 = u_{I+1} - u_I$.

- Higher derivatives:

$$\frac{d^r}{du^r}\mathbf{x}(u) = \frac{n!}{(n-r)!}\mathbf{b}[u^{<n-r>}, \vec{1}^{<r>}]$$

Explicit Representation

$$\mathbf{x}(u) = \sum_{i=0}^{n} \mathbf{d}_i P_i^n(u)$$

The polynomial P_i^n satisfy the following recurrence relation:

$$P_i^n(u) = (1 - t_{i+1}^n) P_i^{n-1}(u) + t_i^n P_{i-1}^{n-1}(u)$$

with base cases

$$P_0^1(u) = \frac{U_1^0 - u}{|U|} = \frac{u_{I+1} - u}{u_{I+1} - u_I} \text{ and } P_1^1(u) = \frac{u - U_0^0}{|U|} = \frac{u - u_I}{u_{I+1} - u_I}$$

Proof:

$$\mathbf{x}(u) = \sum_{i=0}^{n-1} \mathbf{d}_i^1 P_i^{n-1}(u) = \sum_{i=0}^{n-1} (1 - t_{i+1}^n) \mathbf{d}_i P_i^{n-1}(u) + \sum_{i=0}^{n-1} t_{i+1}^n \mathbf{d}_{i+1} P_i^{n-1}(u)$$

$$= \sum_{i=0}^{n} (1 - t_{i+1}^n) \mathbf{d}_i P_i^{n-1}(u) + \sum_{i=0}^{n} t_i^n \mathbf{d}_i P_{i-1}^{n-1}(u)$$

where $P_n^{n-1}(u) \equiv 0(u)$ and $P_{-1}^{n-1}(u) \equiv 0(u)$.

Outline

B-Spline Curves

- A B-spline curve is defined by
  - the degree n of each curve segment,
  - the knot sequence $u_0, \ldots, u_K$, consisting of $K+1$ knots $u_i \leqslant u_{i+1}$.
  - the control polygon $\mathbf{d}_0, \ldots, \mathbf{d}_L$ with $L = K - n + 1$.

  $\rightarrow$ Example 8.3
- Each knot may be repeated in the knot sequence up to n times. (Why n?)
- Different de Boor algorithm for each curve segment
- The valid intervals are $[u_{n-1}, u_n], [u_n, u_{n+1}], \ldots, [u_{K-n}, u_L]$ (Why?)
- The *domain* of the B-spline curve is $[u_{n-1}, u_L] = [u_{n-1}, u_n] \cup [u_n, u_{n+1}] \cup \cdots \cup [u_{K-n}, u_L]$
- Examples: Figure 8.3 (a) (b) (c)

B-Spline Curve in Bézier Form

- Each segment is a polynomial $\rightarrow$ Can be expressed in Bézier form
- For the segment defined over $U = [u_I, u_{I+1}]$,
  1. Evaluate its blossom $\mathbf{b}^U[u^{<n>}]$.
  2. Then the Bézier points $\{\mathbf{b}_k^U\}_{k=0}^n$ are defined as

$$\mathbf{b}_k^U = \mathbf{b}^U[u_I^{<n-k>}, u_{I+1}^{<k>}]$$

    - See (4.11) on p.52
    - Figure 8.4 & Figure 8.5

B-Spline Curve in Bézier Form: Example

- Quadratic ($n = 2$)
- Uniform knots $(0, 1, 2, 3)$ with $u_I = 1$ and $u_{I+1} = 2$

1. Find the blossom

$$\begin{aligned}\mathbf{b}[t_1, t_2] =& (2 - t_1)\mathbf{b}[t_1, 1] + (t_1 - 1)\mathbf{b}[t_1, 2] \\ =& (2 - t_1)\left(\frac{2 - t_2}{2}\mathbf{b}[0, 1] + \frac{t_2}{2}\mathbf{b}[1, 2]\right) \\ &+ (t_1 - 1)\left(\frac{3 - t_2}{2}\mathbf{b}[1, 2] + \frac{t_2 - 1}{2}\mathbf{b}[2, 3]\right)\end{aligned}$$

2. Find the Bézier points

$$\mathbf{c}_0 := \mathbf{b}[1, 1] = \frac{1}{2}\mathbf{b}[0, 1] + \frac{1}{2}\mathbf{b}[1, 2]$$

$$\mathbf{c}_1 := \mathbf{b}[1, 2] = \mathbf{b}[1, 2]$$

$$\mathbf{c}_2 := \mathbf{b}[2, 2] = \frac{1}{2}\mathbf{b}[1, 2] + \frac{1}{2}\mathbf{b}[2, 3]$$

Evaluation

- Steps to evaluate $\mathbf{d}(u)$ with $u \in [u_{n-1}, u_{K-n+1}]$.
    1. Find the interval $U = [u_I, u_{I+1})$ that contains u.
    2. Find the $n+1$ control points that are relevant for the interval U. They are, using the global numbering, given by $\mathbf{d}_{I-n+1}, \ldots, \mathbf{d}_{I+1}$.
    3. Renumber them as $\mathbf{d}_0, \ldots, \mathbf{d}_n$ and evaluate using the de Boor algorithm.
- Each curve segment
    - has the local convex hull property (Figure 8.6) and
    - is only affected by $n+1$ control points $\rightarrow$ *local control property* (Figure 8.7)

Outline

Knot Insertion

- For a B-spline curve segment of degree n defined over an interval $U = [u_I, u_{I+1}]$...
  - Defined by all blossom values $\mathbf{b}[U_i^{n-1}], i = 0, \ldots, n$
  - Each n-tuple of successive knots U_i^{n-1} contains at least one of the endpoints of $U = [u_I, u_{I+1}]$.
- Let's split $U = [u_I, u_{I+1}]$ into two segments $[u_I, \hat{u}]$ and $[\hat{u}, u_{I+1}]$ by inserting a new knot $\hat{u}$ inbetween. What happens?
  $\rightarrow$ The curve segment is split into two. (With $\hat{U}_i^{n-1}, i = 0, \ldots, n+1$ are n-tuples of successive knots containing at least one of the endpoints of $U = [u_I, u_{I+1}]$)
  - One defined over $[u_I, \hat{u}]$ and by the control points $\{\mathbf{b}[\hat{U}_i^{n-1}]\}_{i=0}^{n}$ and
  - the other defined over $[\hat{u}, u_{I+1}]$ and by the control points $\{\mathbf{b}[\hat{U}_i^{n-1}\}_{i=1}^{n+1}$.
  - Example 8.4
  - Figure 8.8
- Chaikin's algorithm (Figure 8.9)

Outline

Degree Elevation

- Can be done in (almost) the same way as for Bézier curves.
- Since the differentiability (at knots) is determined by the knot multiplicities, (Sec '8.7 Smoothness') we need to increase the multiplicity of every knot by one.
- Degree $n+1$ blossom $\hat{\mathbf{b}}$ in terms of degree n blossoms $\mathbf{b}$:

$$\hat{\mathbf{b}}[\,V^{(n+1)}] = \frac{1}{n+1}\left(\mathbf{b}[\,V^{(n+1)}|v_1] + \cdots + \mathbf{b}[\,V^{(n+1)}|v_{n+1}]\right)$$

  - $V^n := v_1, \ldots, v_{n+1}$
  - $V^n|v_i := v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}$: the sequence V^n with v_i removed.

Degree Elevation: Example 8.5

before	$u_0 = u_1 = u_2$	u_3	u_4	u_5	$\dots$
after	$\hat{u}_0 = \hat{u}_1 = \hat{u}_2 = \hat{u}_3$	$\hat{u}_4 = \hat{u}_5$	$\hat{u}_6 = \hat{u}_7$	$\hat{u}_8 = \hat{u}_9$	$\dots$

- The interval $[u_4, u_5]$ corresponds to $[\hat{u}_7, \hat{u}_8]$.
- $\mathbf{d}_4$: The blossom before degree elevation
- $\mathbf{d}_7$: The blossom after degree elevation
- $\{\hat{\mathbf{d}}_4, \hat{\mathbf{d}}_5, \hat{\mathbf{d}}_6, \hat{\mathbf{d}}_7\}$: Control points after degree elevation
-

$$\begin{aligned}
\hat{\mathbf{d}}_4 &:= \hat{\mathbf{d}}_7[\hat{u}_4, \hat{u}_5, \hat{u}_6, \hat{u}_7] \\
&= \frac{1}{4}\left(\mathbf{d}_4[\hat{u}_4, \hat{u}_5, \hat{u}_6] + \mathbf{d}_4[\hat{u}_4, \hat{u}_5, \hat{u}_7] + \mathbf{d}_4[\hat{u}_4, \hat{u}_6, \hat{u}_7] + \mathbf{d}_4[\hat{u}_5, \hat{u}_6, \hat{u}_7]\right) \\
&= \frac{1}{2}\left(\mathbf{d}_4[u_3, u_3, u_4] + \mathbf{d}_4[u_3, u_4, u_4]\right)
\end{aligned}$$

Outline

Greville Abscissae

- Where should we locate the x-coordinates of control points such that the x-coodrinates of the curve change the same as u?

$$\xi_i = \frac{1}{n}(u_i + \cdots + u_{i+n-1})$$

- Figure 8.10

Outline

Smoothness

- A B-spline is a piecewise polynomial curve. Then what is the smoothness at each knot?
- If a knot $\hat{u}$ is of multiplicity r, then a B-spline curve of degree n has smoothness C^{n-r} at $\hat{u}$.
- Figure 8.13 & Figure 8.15

Outline

B-Splines

- Given ordered knot sequence $\{u_0, \ldots, u_{n-1}, \ldots, u_{K-n+1}, \ldots, u_K\}$,
  - B-spline curves of degree n can be defined over $[u_{n-1}, u_L]$ ($L = K - n + 1$, p.126)
  - The dimension of the linear space formed by all piecewise polynomials over $[u_{n-1}, u_L]$=?
    =# of control points =$L + 1$
  - Control points: $\mathbf{b}[u_0, \ldots, u_{n-1}], \ldots, \mathbf{b}[u_{K-n+1}, \ldots, u_K]$

B-Splines (cont'd)

- B-splines in canonical form: $\mathbf{x}(u) = \sum_{j=0}^{L} \mathbf{d}_j N_j^n(u)$
  - $\mathbf{d}_j = \mathbf{b}[u_j, \ldots, u_{j+n-1}]$
  - Figure 8.17
- Local support: $N_j^n(u) \neq 0$ only if $u \in [u_{j-1}, u_{j+n}]$.
- $\{N_j^n\}$ are linearly independent.
- Partition of unity: $\sum_{j=0}^{L} N_j^n(u) = 1(u)$ for $u \in [u_{n-1}, u_L]$.
- Recursive relation

$$N_j^n(u) = \frac{u - u_{j-1}}{u_{j+n-1} - u_{j-1}} N_j^{n-1}(u) + \frac{u_{j+n} - u}{u_{j+n} - u_j} N_{j+1}^{n-1}(u)$$

  with

$$N_j^0(u) = \begin{cases} 1 & \text{if } u_{j-1} \leqslant u < u_j \\ 0 & \text{else.} \end{cases}$$

  - Figure 8.18

Outline

B-Spline Basics

- n: (maximal) degree of each polynomial segment
- K: # of intervals
- $L + 1$: # of control points ($L = K - n + 1$)
- knot sequence: $\{u_0, \ldots, u_K\}$
- control points: $\mathbf{d}_0, \ldots, \mathbf{d}_L$ with $L = K - n + 1$
- Domain: Curve is only defined over $[u_{n-1}, \ldots, u_L]$.
- Greville abscissae: $\xi_i = \frac{1}{n}(u_i + \cdots + u_{i+n-1})$.
- Support: N_i^n is nonnegative over $[u_{i-1}, u_{i+n}]$.
- Knot insertion: To insert $u_I \leqslant u < u_{I+1}$, first find new Greville abscissae $\hat{\xi}_i$, then set $d_i = P(\hat{\xi}_i)$.

B-Spline Basics (cont'd)

- de Boor algorithm: Given $u_I \leqslant u < u_{I+1}$, renumber the relevant control points $\mathbf{d}_{I-n+1}, \dots, \mathbf{d}_{I+1}$ as $\mathbf{d}_0, \dots, \mathbf{d}_n$ and then set

  $$\mathbf{d}_i^k(u) = (1 - \alpha_i^k)\mathbf{d}_i^{k-1}(u) + \alpha_i^k \mathbf{d}_{i+1}^{k-1}(u)$$

  with

  $$\alpha_i^k := \frac{u - u_{I-n+k+i}}{u_{I+i+1} - u_{I-n+k+i}}$$

  for $k = r+1, \dots, n$ and $i = 0, \dots, n-k$. Here, r denots the multiplicity of u.
- Mansfield, de Boor, Cox recursion:

  $$N_j^n(u) = \frac{u - u_{j-1}}{u_{j+n-1} - u_{j-1}} N_j^{n-1}(u) + \frac{u_{j+n} - u}{u_{j+n} - u_j} N_{j+1}^{n-1}(u).$$

B-Spline Basics (cont'd)

- Derivative:

$$\frac{d}{du}N_j^n(u) = \frac{n}{u_{n+j-1} - u_{j-1}} N_j^{n-1}(u) - \frac{n}{u_{j+n} - u_j} N_{j+1}^{n-1}(u).$$

- Derivative of B-spline curve:

$$\frac{d}{du}\mathbf{x}(u) = n \sum_{i=1}^{L-1} \frac{\Delta \mathbf{d}_{i-1}}{u_{n+i-1} - u_{i-1}} N_i^{n-1}(u).$$

- Degree elevation:

$$N_i^n(u) = \frac{1}{n+1} \sum_{j=i-1}^{n+i} N_i^{n+1}(u; u_j),$$

where $N_i^{n+1}(u; u_j)$ is defined over the original knot sequence except that the knot u_j has its multiplicity increased by one.