

Topics in Computer Graphics

Chap 6: Bézier Curve Topics

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Degree Elevation

- ▶ What if we want to get more flexibility while keeping the shape of the current curve? → *Degree elevation* required
- ▶ Monomial form case?
- ▶ Can be used to make several curves have the same degree
- ▶ Useful for data transfer between different CAD/CAM or graphics systems
- ▶ **Figure 6.1**

Degree Elevation: Algebraic Proof

$$(1-t)B_i^n(t) = \frac{n+1-i}{n+1}B_i^{n+1}(t)$$

$$tB_i^n(t) = \frac{i+1}{n+1}B_{i+1}^{n+1}(t)$$

$$B_i^n(t) = \frac{n+1-i}{n+1}B_i^{n+1}(t) + \frac{i+1}{n+1}B_{i+1}^{n+1}(t)$$

→

$$\begin{aligned}\mathbf{x}(t) &= (1-t)\mathbf{x}(t) + t\mathbf{x}(t) \\ &= \sum_{i=0}^n \frac{n+1-i}{n+1} \mathbf{b}_i B_i^{n+1}(t) + \sum_{i=0}^n \frac{i+1}{n+1} \mathbf{b}_i B_{i+1}^{n+1}(t) \\ &= \sum_{i=0}^{n+1} \left(\frac{i}{n+1} \mathbf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right) \mathbf{b}_i \right) B_i^{n+1}(t) \\ &=: \sum_{i=0}^{n+1} \mathbf{b}_i^{(1)} B_i^{n+1}(t)\end{aligned}$$

Degree Elevation: Proof Using Blossoms

$$\mathbf{b}^{(1)}[t_1, \dots, t_{n+1}] = \frac{1}{n+1} \sum_{j=1}^{n+1} \mathbf{b}[t_1, \dots, t_{n+1} | t_j]$$

- ▶ $\mathbf{b}[t_1, \dots, t_{n+1} | t_j]$: t_j is omitted from $\mathbf{b}[t_1, \dots, t_{n+1}]$.
- ▶ Why? → Definition of a blossom

$$\begin{aligned} \mathbf{b}_i^{(1)} &= \mathbf{b}^{(1)}[0^{<n+1-i>}, 1^{<i>}] \\ &= \frac{1}{n+1} (i\mathbf{b}_{i-1} + (n+1-i)\mathbf{b}_i) \end{aligned}$$

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Repeated Degree Elevation

- ▶ Degree elevation: $\mathbf{P} \rightarrow \mathcal{E}\mathbf{P}$
- ▶ After r degree elevation, $\mathcal{E}^r\mathbf{P}$ has the vertices $\{\mathbf{b}_i^{(r)}\}_{i=0}^{n+r}$ where

$$\mathbf{b}_i^{(r)} = \sum_{j=0}^n \mathbf{b}_j \binom{n}{j} \frac{\binom{r}{i-j}}{\binom{n+r}{i}}$$

- ▶ $\lim_{r \rightarrow \infty} \mathcal{E}^r\mathbf{P} = \mathcal{B}\mathbf{P} \quad \leftarrow \quad \lim_{i/(n+r) \rightarrow t} \frac{\binom{r}{i-j}}{\binom{n+r}{i}} = t^j(1-t)^{n-j}$
 - ▶ $\mathcal{E}^r\mathbf{P}$ converges to the Bézier curve $\mathcal{B}\mathbf{P}$.
 - ▶ But converges too slowly.

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The Variation Diminishing Property

- ▶ The curve \mathcal{BP} has no more intersections with any plane than does the polygon \mathbf{P} .
- ▶ Proof
 1. Degree elevation is a piecewise linear interpolation.
 2. Piecewise linear interpolation is variation diminishing. (Sec 3.2)
 - Each $\mathcal{E}^r\mathbf{P}$ has fewer intersections than $\mathcal{E}^{r-1}\mathbf{P}$.
 3. Since $\mathcal{E}^r\mathbf{P}$ converges to the curve, the proof is done.
- ▶ A convex polygon generates a convex curve.
How about the inverse?

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Degree Reduction

- ▶ Problem: Can we write a given curve of degree $n + 1$ as one of degree n ?
- ▶ In general, exact degree reduction is not possible.
→ approximate method
- ▶ New problem: Given a Bézier curve with control vertices $\{\mathbf{b}_i^{(1)}\}_{i=0}^{n+1}$, can we find a Bézier curve with control vertices $\{\mathbf{b}_i\}_{i=1}^n$ that approximates the first curve in a “reasonable” way?

$$\begin{bmatrix} 1 & & & & & & & \\ & * & & & & & & \\ & & * & & & & & \\ & & & * & & & & \\ & & & & * & & & \\ & & & & & \ddots & & \\ & & & & & & * & \\ & & & & & & & * \\ & & & & & & & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b}_0 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0^{(1)} \\ \vdots \\ \mathbf{b}_{n+1}^{(1)} \end{bmatrix}$$

Degree Reduction (cont'd)

- ▶ $MB = B^{(1)}$, $M \in \mathbb{R}^{(n+2) \times (n+1)}$
→ Not solvable.
- ▶ Can be solved after converted to

$$(M^T M) B = M^T B^{(1)}$$

- ▶ *Normal equation*
- ▶ Guarantees that B is optimal in a least square sense.
- ▶ Problem?
 $\mathbf{b}_0 = \mathbf{b}_0^{(1)}$ and $\mathbf{b}_n = \mathbf{b}_{n+1}^{(1)}$ are not guaranteed.

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Nonparametric Curves

- ▶ Functional curves: $y = f(x)$
- ▶ How can we express a functional curve in parametric form?

$$\mathbf{b}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ f(t) \end{bmatrix}$$

- ▶ If f is a polynomial,

$$f(t) = b_0 B_0^n(t) + \cdots + b_n B_n^n(t), \quad b_i \in \mathbb{R} \text{ for } i = 0, \dots, n.$$

- ▶ How to find the control polygon?

$$\mathbf{b}(t) = \sum_{j=0}^n \begin{bmatrix} j/n \\ b_j \end{bmatrix} B_j^n(t)$$

→ The control polygon is given by the points $\{(j/n, b_j)\}_{j=0}^n$.

- ▶ $f(t)$ is called a *Bézier function*. (Figure 6.3)
- ▶ $\{(j/n, b_j)\}_{j=0}^n$ are called *Bézier ordinates*.
- ▶ $\{j/n\}_{j=0}^n$ are called the *abscissae*.

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- ▶ For a 2D Bézier curve, plot each coordinate function as a Bézier function.
- ▶ **Figure 6.4**

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Integrals

- ▶ The derivative of a polynomial function $b(t)$:

$$\dot{b}(t) = n \sum_{i=0}^{n-1} \Delta b_i B_i^{n-1}(t)$$

- ▶ The indefinite integral (or antiderivative) of a polynomial $b(t)$ in Bernstein form defined by the ordinates $\{b_i\}_{i=0}^n$ is the polynomial $B(t)$ defined by the ordinates

$$\frac{1}{n+1} [c, c + b_0, c + b_0 + b_1, \dots, c + \sum_{i=0}^n b_i]$$

- ▶ $\int_0^1 b(t) dt = B(1) - B(0) = \frac{1}{n+1} \sum_{i=0}^n b_i$

- ▶ $\int_0^1 B_j^n(x) dx = \frac{1}{n+1}$ for any j

- ▶ Can be obtained by setting $b_i = \delta_{i,j}$.
- ▶ All basis functions B_j^n have the same definite integral.

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The Bézier Form of a Bézier Curve

- ▶ The basis functions Bézier used:

$$\mathbf{b}^n(t) = \sum_{j=0}^n \mathbf{c}_j F_j^n(t)$$

- ▶ F_j^n are polynomials that obey the recursion

$$F_i^n(t) = (1-t)F_i^{n-1}(t) + tF_{i-1}^{n-1}(t)$$
$$F_0^0(t) = 1(t), \quad F_{r+1}^r(t) = 0(t), \quad F_{-1}^r(t) = 1(t)$$

- ▶ Explicit form: $F_i^n = \sum_{j=i}^n B_j^n$

- ▶ $F_0^n \equiv 1(t)$ for all n

→ Not a barycentric combination!

→ How can it be defined then?

The Bézier Form of a Bézier Curve (cont'd)

- ▶ $\mathbf{c}_0 = \mathbf{b}_0 \in \mathbb{E}^3$ and $\mathbf{c}_j = \Delta \mathbf{b}_{j-1} \in \mathbb{R}^3$ for $j > 0$
- ▶ $F_i^n(t)$ is NOT symmetric w.r.t. t and $1 - t$.
- ▶ The value of $\mathbf{x}(1)$ is the sum of all errors in the \mathbf{c}_i .

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The Weierstrass Approximation Theorem

1. $\mathbf{c}(t)$: parametric curve defined over $[0, 1]$
2. For some fixed n , sample $\mathbf{c}(t)$ at parameter values i/n .
3. Define a Bézier curve $\mathbf{x}_n(t)$ by control points $\{\mathbf{c}(i/n)\}_{i=0}^n$:

$$\mathbf{x}_n(t) = \sum_{i=0}^n \mathbf{c} \left(\frac{i}{n} \right) B_i^n(t)$$

“ $\mathbf{x}(t)$ is the n^{th} degree Bernstein-Bézier approximation to \mathbf{c} .”

4. Weierstrass approximation theorem:

$$\lim_{n \rightarrow \infty} \mathbf{x}_n(t) = \mathbf{c}(t)$$

- ▶ Not practical (too large n required)
- ▶ Every curve may be approximated arbitrarily closely by a polynomial curve.

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Formulas for Bernstein Polynomials

- ▶ A Bernstein polynomial is defined by

$$B_i^n(t) = \begin{cases} \binom{n}{i} t^i (1-t)^{n-i} & \text{if } i \in [0, n], \\ 0 & \text{else.} \end{cases}$$

- ▶ Relationship with the power basis $\{t^i\}$

$$t^i = \sum_{j=i}^n \frac{\binom{j}{i}}{\binom{n}{i}} B_j^n(t)$$

and

$$B_i^n(t) = \sum_{j=i}^n (-1)^{j-i} \binom{n}{j} \binom{j}{i} t^j.$$

Formulas for Bernstein Polynomials (cont'd)

- ▶ Recursion:

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

- ▶ Subdivision:

$$B_i^n(ct) = \sum_{j=0}^n B_i^j(c) B_j^n(t)$$

- ▶ Derivative:

$$\frac{d}{dt} B_i^n(t) = n [B_{i-1}^{n-1}(t) - B_i^{n-1}(t)]$$

- ▶ Integral:

$$\int_0^t B_i^n(x) dx = \frac{1}{n+1} \sum_{j=i+1}^{n+1} B_j^{n+1}(t)$$

$$\int_0^1 B_i^n(x) dx = \frac{1}{n+1}$$

Formulas for Bernstein Polynomials (cont'd)

- ▶ Degree elevation formulas:

$$(1-t)B_i^n(t) = \frac{n+1-i}{n+1}B_i^{n+1}(t)$$

$$tB_i^n(t) = \frac{i+1}{n+1}B_{i+1}^{n+1}(t)$$

$$B_i^n(t) = \frac{n+1-i}{n+1}B_i^{n+1}(t) + \frac{i+1}{n+1}B_{i+1}^{n+1}(t)$$

- ▶ Product:

$$B_i^m(u)B_j^n(u) = \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}}B_{i+j}^{m+n}(u)$$