

Topics in Computer Graphics
Chap 5: The Bernstein Form of a Bézier Curve
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University of Seoul
School of Computer Science
Minho Kim

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Bernstein Polynomials

- ▶ Useful when expressing Bézier curves *explicitly*.
- ▶ Definition:

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i},$$

where the *binomial coefficient* is defined as

$$\binom{n}{i} := \begin{cases} \frac{n!}{i!(n-i)!} & \text{if } 0 \leq i \leq n \\ 0 & \text{else.} \end{cases}$$

- ▶ Figure 5.1

Bernstein Polynomials: Properties

- ▶ Recursive relation

$$B_i^n(t) = (1 - t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$

with

$$B_0^0(t) \equiv 1(t) \quad \text{and} \quad B_j^n(t) \equiv 0(t) \quad \text{for } j \notin 0, \dots, n.$$

- ▶ *Partition of unity*

$$\sum_{j=0}^n B_j^n(t) \equiv 1(t).$$

* Binomial theorem

Bézier Curves Expressed by Bernstein Polynomials

$$\mathbf{b}(t) = \mathbf{b}[t^{\langle t \rangle}] = \sum_{i=0}^n \mathbf{b}_i B_i^n(t)$$

Can be proved by Leibniz formula

- ▶ Intermediate de Casteljau point

$$\begin{aligned} \mathbf{b}_i^r(t) &= \mathbf{b}[0^{\langle n-r-i \rangle}, t^{\langle r \rangle}, 1^{\langle i \rangle}] && (4.10) \\ &= \sum_{j=0}^r \mathbf{b}[0^{\langle n-r-i \rangle}, 1^{\langle i \rangle}, 0^{\langle r-j \rangle}, 1^{\langle j \rangle}] B_j^r(t) \\ &= \sum_{j=0}^r \mathbf{b}[0^{\langle n-i-j \rangle}, 1^{\langle i+j \rangle}] B_j^r(t) = \sum_{j=0}^r \mathbf{b}_{i+j} B_j^r(t) \end{aligned}$$

- ▶ Bézier curve in terms of intermediate points

$$\begin{aligned} \mathbf{b}^n(t) &= \mathbf{b}[t^{\langle n-r \rangle}, t^{\langle r \rangle}] = \sum_{i=0}^{n-r} \mathbf{b}[t^{\langle r \rangle}, 1^{\langle i \rangle}, 0^{\langle n-r-i \rangle}] B_i^{n-r}(t) \\ &= \sum_{i=0}^{n-r} \mathbf{b}_i^r(t) B_i^{n-r}(t) \end{aligned}$$

- ▶ Figure 5.2

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Properties of Bézier Curves

- ▶ Affine invariance
 - ▶ Invariance under affine parameter transformations
 - ▶ Convex hull property (Figure 5.3)
 - ▶ Endpoint interpolation
 - ▶ Symmetry
 - ▶ Invariance under barycentric combinations (Figure 5.4)
 - ▶ Linear precision
 - ▶ Pseudolocal control
- Can be proved algebraically using Bernstein polynomials.

Properties of Bézier Curves (cont'd)

- ▶ Affine invariance
 - ← Any point on a Bézier curve is a barycentric combination of its control points. (Why?)

$$\sum_{i=0}^n B_i^n(t) \equiv 1(t)$$

- ▶ Invariance under affine parameter transformations

$$\sum_{i=0}^n \mathbf{b}_i B_i^n(t) = \sum_{i=0}^n \mathbf{b}_i B_i^n\left(\frac{u-a}{b-a}\right)$$

- ▶ Convex hull property
- ▶ Endpoint interpolation

$$B_i^n(0) = \delta_{i,0} \quad B_i^n(1) = \delta_{i,n}$$

* Kronecker delta function

Properties of Bézier Curves (cont'd)

- Symmetry

$$\sum_{j=0}^n \mathbf{b}_j B_j^n(t) = \sum_{j=0}^n \mathbf{b}_{n-j} B_j^n(1-t) \leftarrow B_j^n(t) = B_{n-j}^n(1-t)$$

- Invariance under barycentric combinations ($\alpha + \beta = 1$)

$$\sum_{j=0}^n (\alpha \mathbf{b}_j + \beta \mathbf{c}_j) B_j^n(t) = \alpha \sum_{j=0}^n \mathbf{b}_j B_j^n(t) + \beta \sum_{j=0}^n \mathbf{c}_j B_j^n(t)$$

→ tensor-product surfaces

- Linear precision

$$\sum_{j=0}^n \frac{j}{n} B_j^n(t) = t$$

- Pseudolocal control $\operatorname{argmax}_{0 \leq t \leq 1} B_j^n(t) = j/n$

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The Derivative of a Bézier Curve

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = n\mathbf{b}[t^{<n-1>}, \vec{1}]$$

- ▶ First interpretation (Figure 5.6)

$$= n \sum_{j=0}^n \mathbf{b}[1^{<j>}, 0^{<n-1-j>}, \vec{1}] B_j^{n-1}(t)$$

$$= n \sum_{j=0}^n \left(\mathbf{b}[1^{<j+1>}, 0^{<n-(j+1)>}] - \mathbf{b}[1^{<j>}, 0^{<n-j>}] \right) B_j^{n-1}(t)$$

$$(\mathbf{b}[\vec{1}, *] = \mathbf{b}[1, *] - \mathbf{b}[0, *])$$

$$= n \sum_{j=0}^{n-1} (\mathbf{b}_{j+1} - \mathbf{b}_j) B_j^{n-1}(t)$$

$$= n \sum_{j=0}^{n-1} \Delta \mathbf{b}_j B_j^{n-1}(t) \quad (\Delta \mathbf{b}_j := \mathbf{b}_{j+1} - \mathbf{b}_j \in \mathbb{R}^3)$$

- ▶ Bézier curve where its coefficients are vectors, not points.
→ *hodograph*

The Derivative of a Bézier Curve (cont'd)

- ▶ Second interpretation

$$\begin{aligned}n\mathbf{b}[t^{\langle n-1 \rangle}, \vec{1}] &= n(\mathbf{b}[t^{\langle n-1 \rangle}, 1] - \mathbf{b}[t^{\langle n-1 \rangle}, 0]) \\ &= n(\mathbf{b}_1^{n-1}(t) - \mathbf{b}_0^{n-1}(t))\end{aligned}\quad (4.10)$$

Higher Derivatives

$$\begin{aligned}\frac{d^r \mathbf{x}(t)}{dt^r} &= \frac{n!}{(n-r)!} \mathbf{b}[t^{<n-r>}, \vec{1}^{<r>}] \\ &= \frac{n!}{(n-r)!} \sum_{j=0}^{n-r} \Delta^r \mathbf{b}_j B_j^{n-r}(t)\end{aligned}$$

where the *iterated forward difference operator* Δ^r is defined recursively as

$$\Delta^r \mathbf{b}_j = \Delta^{r-1} \mathbf{b}_{j+1} - \Delta^{r-1} \mathbf{b}_j$$

of which explicit form is

$$\Delta^r \mathbf{b}_i = \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} \mathbf{b}_{i+j}.$$

Higher Derivatives (cont'd)

$$\frac{d^r}{dt^r} \mathbf{b}^n(0) = \frac{n!}{(n-r)!} \Delta^r \mathbf{b}_0$$
$$\frac{d^r}{dt^r} \mathbf{b}^n(1) = \frac{n!}{(n-r)!} \Delta^r \mathbf{b}_{n-r}$$

- ▶ “The r th derivative of a Bézier curve at an endpoint depends only the $r + 1$ Bézier points near (and including) that endpoint.”
- ▶ Figure 5.7

Bézier Curves in Monomial Form

$$\mathbf{b}^n(t) = \sum_{j=0}^n \binom{n}{j} \Delta^j \mathbf{b}_0 t^j$$

- ▶ Can be derived from Taylor series

$$\mathbf{x}(t) = \sum_{j=0}^n \frac{1}{j!} \mathbf{x}^{(j)}(0) t^j$$

where

$$\mathbf{x}^{(j)}(t) = \frac{n!}{(n-j)!} \sum_{i=0}^{n-j} \Delta^j \mathbf{b}_i B_i^{n-j}(t).$$

- ▶ Numerically unstable

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Domain Changes and Subdivision

- ▶ “Given a Bézier curve defined over $[0, 1]$, what are the control points $\{c_i\}$ of the part defined over $[0, c]$?” (See (4.11) on p.52.)

$$\begin{aligned}\mathbf{x}_c(t) &:= \sum_{j=0}^n \mathbf{c}_j B_j^n(t) = \mathbf{x}(ct) = \sum_{j=0}^n \mathbf{b}_j B_j^n(ct) \\ &= \sum_{j=0}^n \mathbf{b}_j \left(\sum_{i=0}^n B_j^i(c) B_i^n(t) \right) \\ &= \sum_{i=0}^n \left(\sum_{j=0}^i \mathbf{b}_j B_j^i(c) \right) B_i^n(t)\end{aligned}\tag{6.22}$$

$$\mathbf{c}_i = \mathbf{b}[0^{<n-i>}, c^{<i>}] = \mathbf{b}_0^i(c)\tag{4.11}$$

→ *Subdivision formula* for Bézier curves
(Figure 5.8)

Extrapolation

Control points $\{\mathbf{d}_i\}$ corresponding to an interval $[1, d]$

$$\mathbf{d}_j = \mathbf{b}[1^{\langle n-j \rangle}, d^{\langle j \rangle}] = \mathbf{b}_{n-j}^j(d)$$

- ▶ Numerically unstable for large d

Subdivision and Applications

- ▶ The control polygon converges to the curve *quickly* as the subdivision is repeated.
- ▶ Applications
 - ▶ Rendering a Bézier curve
 - ▶ Line-planar curve intersection test
 - ▶ **Figure 5.9**

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How to generate complex shapes using Bézier curves?

- ▶ Using high degree curves
 - ▶ Expensive to evaluate
 - ▶ Too smooth to generate complex shapes ([Java applet](#))
- ▶ Using composite curves
 - ▶ Requires points properly positioned for continuity
 - ▶ C^0 continuity: two curves share the joint
 - ▶ G^1 continuity: the directions (not lengths) of tangent vectors at the joint are the same
 - ▶ C^1 continuity: both the directions & lengths of tangent vectors at the joint are the same

Continuity at the Joint

With \mathbf{x}_- defined by $\mathbf{b}_0, \dots, \mathbf{b}_3$ over $[a, b]$ and \mathbf{b}_+ defined by $\mathbf{b}_3, \dots, \mathbf{b}_6$ over $[b, c]$, the following condition should be met for C^1 continuity:

$$\frac{3}{b-a}[\mathbf{b}_3 - \mathbf{b}_2] = \frac{3}{c-b}[\mathbf{b}_4 - \mathbf{b}_3]$$

→ The ratio of the three points $\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ is the same as the ratio of the three parameter values a, b, c .

- ▶ **Figure 5.12 and 5.13**
- ▶ Continuities are guaranteed easily for B-splines. (Chap 8)

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The Matrix Form of a Bézier Curve

A curve of the form

$$\mathbf{x}(t) = \sum_{j=0}^n \mathbf{c}_j C_j(t)$$

can be expressed as

$$\mathbf{x}(t) = [\mathbf{c}_0 \quad \cdots \quad \mathbf{c}_n] \begin{bmatrix} C_0(t) \\ \vdots \\ C_n(t) \end{bmatrix}$$

where

$$\begin{bmatrix} C_0(t) \\ \vdots \\ C_n(t) \end{bmatrix} = M \begin{bmatrix} t^0 \\ \vdots \\ t^n \end{bmatrix} = \begin{bmatrix} m_{00} & \cdots & m_{0n} \\ \vdots & & \vdots \\ m_{n0} & \cdots & m_{nn} \end{bmatrix} \begin{bmatrix} t^0 \\ \vdots \\ t^n \end{bmatrix}$$

M : basis transformation between the basis polynomial $C_i(t)$ and the *monomial basis* t^i

The Matrix Form of a Bézier Curve (cont'd)

If $C_i(t) = B_i^n(t)$, (Bernstein polynomials),

$$m_{ij} = (-1)^{j-i} \binom{n}{j} \binom{j}{i}$$

Example (cubic case)

$$M = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ▶ Again, monomial forms are not numerically stable.