Solution for homework #3-1

April 23, 2012

- Excercises 3-1

17 (You don't have to derive the following formula, but just need to show one example that satisfies it.)
We want a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$A^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

except $A = O$. From the entry $A_{1,2}$, we can consider two cases:

(a) $a + d \neq 0$
Then,

$$\begin{aligned} A_{1,2} &= b(a+d) = 0 \to b = 0 \\ A_{2,1} &= c(a+d) = 0 \to c = 0 \\ A_{1,1} &= a^2 + bc = 0 \to a = 0 \\ A_{2,2} &= d^2 + bc = 0 \to d = 0 \end{aligned}$$

Since this is the case of $A = O$, we exclude this case.

(b) $b \neq 0$
Then,

$$\begin{aligned} A_{1,2} &= b(a+d) = 0 \to a + d = 0 \\ A_{2,1} &= c(a+d) = 0 \to c \text{ can be arbitrary since } a + d = 0. \\ A_{1,1} &= a^2 + bc = 0 \to c = -a^2/b \\ A_{2,2} &= d^2 + bc = 0 \to c = -d^2/b = -a^2/b \end{aligned}$$

Suming up, letting $a = s$ and $b = t \neq 0$, A is the form of

$$A = \begin{bmatrix} s & t \\ -s^2/t & -s \end{bmatrix}, \quad t \neq 0.$$

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$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \text{ and } A_{22} = \begin{bmatrix} 4 \end{bmatrix}$$

and

$$B_{11} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, B_{12} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, B_{21} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \text{ and } B_{22} = \begin{bmatrix} -1 \end{bmatrix}.$$

Then,

$$AB = \begin{bmatrix} IB_{11} + A_{12}B_{21} & IB_{12} + A_{12}B_{22} \\ OB_{11} + A_{22}B_{21} & OB_{12} + A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} + \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} & B_{12} - A_{12} \\ 4B_{21} & -4 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 6 \\ 3 & 3 & 4 \\ 4 & 4 & 4 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \\ -2 \\ -4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 0 \\ 2 & 3 & 6 & -1 \\ 3 & 3 & 4 & -2 \\ 4 & 4 & 4 & -4 \end{bmatrix}.$$

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$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}.$$

Let's assume that

$$A^n = \begin{bmatrix} 1 & 2^{n-1} \\ 0 & 1 \end{bmatrix}$$

and prove that this actually is the case by mathematical induction.

(a) (Basis step) Base case $n = 1$:

$$A^1 = \begin{bmatrix} 1 & 2^{1-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore the base case is true.

(b) (Induction hypothesis)
Assume that our formula (A^n) holds for $n = k \geqslant 1$.

(c) (Induction step)
From the induction hypothesis,

$$A^{k+1} = A^k A = \begin{bmatrix} 1 & 2^{k-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2^k \\ 0 & 0 \end{bmatrix}.$$

Therefore our forumla holds for all $n \geqslant 1$.