Solution for homework #1

April 5, 2012

- Problem 6(c) By the definition of the cross product,

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix},$$

  therefore

$$\begin{aligned} \|\boldsymbol{u} \times \boldsymbol{v}\|^2 =& (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ =& (u_2v_3)^2 + (u_3v_2)^2 + (u_3v_1)^2 + (u_1v_3)^2 + (u_1v_2)^2 + (u_2v_1)^2 \\ & - 2u_2u_3v_2v_3 - 2u_1u_3v_1v_3 - 2u_1u_2v_1v_2. \end{aligned}$$

  On the other hand,

$$\begin{aligned} & \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2 \\ =& (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ =& \cancel{(u_1v_1)^2 + (u_2v_2)^2 + (u_3v_3)^2} + (u_1v_2)^2 + (u_1v_3)^2 + (u_2v_1)^2 + (u_2v_3)^2 + (u_3v_1)^2 + (u_3v_2)^2 \\ & - \left\{ \cancel{(u_1v_1)^2 + (u_2v_2)^2 + (u_3v_3)^2} + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3 \right\}. \end{aligned}$$

  Therefore the equality holds.

- Excercises 1.2

  54 No. A counterexample:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

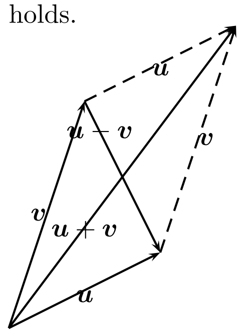
  56 (a)

$$\begin{aligned} \|\boldsymbol{u} + \boldsymbol{v}\|^2 + \|\boldsymbol{u} - \boldsymbol{v}\|^2 &= (\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) \\ &= (\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\boldsymbol{u} \cdot \boldsymbol{v}) + (\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\boldsymbol{u} \cdot \boldsymbol{v}) \\ &= 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2. \end{aligned}$$

(b) In the following figure, the equality

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 + \|\boldsymbol{u}-\boldsymbol{v}\|^2 = 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2$$

holds.



57

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2 &= (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) - (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v}) \\ &= (\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\boldsymbol{u}\cdot\boldsymbol{v}) - (\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\boldsymbol{u}\cdot\boldsymbol{v}) \\ &= 4\boldsymbol{u}\cdot\boldsymbol{v}\end{aligned}$$

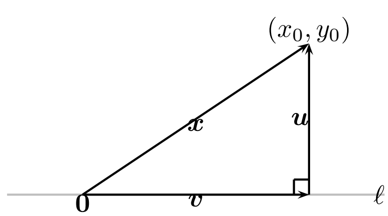
threfore the equality holds.

62 (a) $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{w}0 + 0 = 0.$

(b) $\boldsymbol{u}\cdot(s\boldsymbol{u}+t\boldsymbol{w}) = s(\boldsymbol{u}\cdot\boldsymbol{v}) + t(\boldsymbol{u}\cdot\boldsymbol{w}) = 0.$

- Excercises 1.3

39 See the figure below.



From the line equation

$$ax + by = c,$$

it is clear that vector $\boldsymbol{u}$ in the figure is parallel to $\begin{bmatrix} a \\ b \end{bmatrix}$ therefore we can set

$$\boldsymbol{u} = t\begin{bmatrix} a \\ b \end{bmatrix}$$

for $t \in \mathbb{R}$. Also,

$$\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{u} = \begin{bmatrix} x_0 - ta \\ y_0 - tb \end{bmatrix}.$$

Since $\boldsymbol{v}$ is on the line ℓ, we can plug it into the line equation:

$$a(x_0 - ta) + b(y_0 - tb) = c \to t = \frac{ax_0 + by_0 - c}{a^2 + b^2}$$

Since the distance from (x_0, y_0) to the line ℓ is equal to the length of the vector $\boldsymbol{u}$,

$$\|\boldsymbol{u}\| = |t|\sqrt{a^2 + b^2} = \frac{ax_0 + by_0 - c}{\sqrt{a^2 + b^2}}.$$

40 For a plane, the vector $\boldsymbol{u}$ in the above question becomes

$$\boldsymbol{u} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and $\boldsymbol{v}$ becomes

$$\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{u} = \begin{bmatrix} x_0 - ta \\ y_0 - tb \\ z_0 - tc \end{bmatrix}.$$

Following the same strategy, we can get the correct answer.

41 Note that the distance between two lines is the same as the distance from **any position vector on one line** to the other line.
Let

$$\boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}.$$

Then clearly the position vector

$$\boldsymbol{x} = \begin{bmatrix} c_1/n_1 \\ 0 \end{bmatrix}$$

is on the line

$$\boldsymbol{n} \cdot \boldsymbol{x} = c_1.$$

We can use the equation (3) on p.40:

$$\frac{|n_1(c_1/n_1) - c_2|}{\sqrt{n_1^2 + n_2^2}} = \frac{|c_1 - c_2|}{\|\boldsymbol{n}\|}.$$

45 The normal vector of the plane is

$$\boldsymbol{n} := \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

and the direction vector of the line is

$$\boldsymbol{d} := \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Since

$$\boldsymbol{n} \cdot \boldsymbol{d} = 1 - 2 + 2 = 1 = \|\boldsymbol{n}\|\|\boldsymbol{d}\| \cos\theta = 6\cos\theta.$$

Threfore the angle between the line and the plane is

$$\pi/2 - \cos^{-1}(1/6).$$