Linear Algebra

Chapter 1: Vectors

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Introduction to Linear Algebra

What is *algebra*(대수학:代數學)?

- "Algebra is the branch of mathematics concerning the study of the rules of **operations** and **relations**, and the constructions and concepts arising from them, including **terms**, **polynomials**, **equations** and **algebraic structures**..." (from Wikipedia)
- Elementary algebra
  - operations(연산자): $+$, $-$, $\times$,...
  - relations(관계): $>$, $<$, $=$, $\leq$,...
  - terms(항): '$3x$' and '$4y$' in "$3x + 4y$"
  - polynomials(다항식): $ax^2 + by + cz$
  - algebraic structures(대수적 구조): $\mathbb{Z}$(정수) under addition & multiplication

Introduction to Linear Algebra (cont'd)

What is *linear algebra*?

- "Linear algebra is a branch of mathematics that studies **vector spaces**, also called **linear spaces**, along with **linear functions** that input one vector and output another... (from Wikipedia)
- Vector space is an algebraic structure. $\rightarrow$ operations?

What You've Already Learned in High School...

- 수학관련 교과목: 수학(고1과정), 수학의활용, 수학I, 미적분과통계기본, 수학II, 적분과통계, 기하와벡터
- "7차교육과정" (한국교육과정평가원)
- "대한민국의 고등학교 수학 교과목" (Wikipedia korea)

Related topics:

- 수학(고1과정)
  - 수와 연산 (elementary algebra: 기초대수학)
- 수학I
  - 행렬(matrices)과 그래프
- 기하와 벡터
  - 일차변환(linear transformations)과 행렬(matrices)
  - 공간도형(three-dimensional geometries)과 공간좌표 (three-dimensional coordinates)
  - 벡터(vectors)

Review on Vectors

Let's refresh your memory...

- Definition
- Vector *representation* using coordinates
- Algebra of vectors
- Dot product (점곱) or scalar product (스칼라곱)
  NOTE: Dot(scalar) product *is an* inner product (내적), but *not vice versa*.
- Length of a vector

$\rightarrow$ Watch "Vectors" video on YouTube

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...Enjoy with your friends!

Outline

Vectors

- Direction + magnitude
- **Displacement** from **initial point** (**tail**) to **terminal point** (**head**)
- Notation: $\overrightarrow{AB}$ or $\boldsymbol{v}$
- Position vectors: 1-to-1 correspondence between **points** on the plane and (2D) **vectors**
- Points $\neq$ vectors
- **Representation** $\rightarrow$ Why?
  - **Coordinate system** (e.g., Cartesian, polar)
  - Components
  - **Ordered** pair of real numbers
  - Notation:

    Row vector $[x, y]$ or column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$

    cf) Notation for points: $P = (x, y)$
- Zero vector ($\mathbf{0}$)
- (Real) vector space $\mathbb{R}^2$: “the set of all vectors with two components”

Vector Arithmetic

Let $\boldsymbol{u} := [u_1, u_2]$ and $\boldsymbol{v} := [v_1, v_2]$.
($\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$)

- Addition: $\boldsymbol{u} + \boldsymbol{v} = [u_1 + v_1, u_2 + v_2]$
  - 'Head-to-tail rule' or 'parallelogram rule'
- Scalar multiplication: $c\boldsymbol{u} = c[v_1, v_2] = [cv_1, cv_2]$
  - **Scalar**: constant
- Subtraction: $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v}) = [u_1 - v_1, u_2 - v_2]$

Higher Dimensional Vectors

Vectors in $\mathbb{R}^n$

- **Ordered n-tuples** of real numbers
- $\boldsymbol{v} = [v_1, \cdots, v_n] = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$
- Geometric method doesn't work anymore
  $\rightarrow$ We need algebraic methods using n-tuples.

Algebraic Properties of Vectors in $\mathbb{R}^n$

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

a. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ (commutativity)

b. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$ (associativity)

c. $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$

d. $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$

e. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$ (distributivity)

f. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$ (distributivity)

g. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$

h. $1\boldsymbol{u} = \boldsymbol{u}$

$\rightarrow$ Try to prove them yourselves!

Linear Combination

Definition

A vector $\boldsymbol{v}$ is a **linear combination** of **vectors** $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k$ if there are scalars $c_1, c_2, \cdots, c_k$ such that

$$\boldsymbol{v} = \sum_{j=1}^{k} c_j \boldsymbol{v}_j = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \cdots + c_k \boldsymbol{v}_k.$$

The scalars $c_1, c_2, \cdots, c_k$ are called the **coefficients** of the linear combination.

- Do such scalars $c_1, c_2, \cdots, c_k$ always exist for any $\boldsymbol{v}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k$?
- How many vectors do we need (in $\mathbb{R}^2$)? I.e., how large should k be (in $\mathbb{R}^2$)?
- Is it enough just to have sufficiently many vectors?
- How is the situation different between $\mathbb{R}$ and $\mathbb{R}^2$?
- **Coordinate axes** and **coordinate grid**

Outline

Dot Product

Can we multiply a vector with another vector?
$\to$ **Dot (scalar) product** and **cross product**

Definition

If

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

then the **dot product** $\boldsymbol{u} \cdot \boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{j=1}^{n} u_j v_j = u_1 v_1 + \cdots + u_n v_n \in \mathbb{R}.$$

- The dot product is one of the more general **inner product.** (Chap. 7)

Properties of Dot Product

Theorem 1.2

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

a. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$ (commutativity)

b. $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ (distributivity)

c. $(c\boldsymbol{u}) \cdot \boldsymbol{v} = c(\boldsymbol{u} \cdot \boldsymbol{v})$

d. $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$ and $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ iff (if and only if) $\boldsymbol{u} = \mathbf{0}$.

$\rightarrow$ Try to prove them yourselves!

- $(\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) =?$

Length

Definition

The **length** (or **norm**) of a vector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is the non-negative scalar $\|\boldsymbol{v}\|$ defined by

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$$

Theorem 1.3

Let $\boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

a. $\|\boldsymbol{v}\| = 0$ iff $\boldsymbol{v} = \mathbf{0}$

b. $\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$

Unit Vectors

- Vectors with length 1
- **Normalization**: to make the length of a vector 1
  $\boldsymbol{v} \to (1/\|\boldsymbol{v}\|)\boldsymbol{v}$
- **Standard unit vectors**
  Examples:
  $$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

More on Length

Theorem 1.4: The Cauchy-Schwarz inequality

For all vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$,

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|$$

Theorem 1.5: The triangle inequality

For all vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$,

$$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

Distance

Definition

The **distance** $d(\boldsymbol{u}, \boldsymbol{v})$ between (position) vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ is defined by

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$$

Angles

Definition: Angle

For nonzero vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$,

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

Definition: Orthogonality

Two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ are **orthogonal** to each other ($\boldsymbol{u} \perp \boldsymbol{v}$) if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

- Zero vector

Theorem 1.6: Pythagoras' Theorem

For all vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$,

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$

iff $\boldsymbol{u} \perp \boldsymbol{v}$.

Projections

Definition

If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{u} \neq \mathbf{0}$, the the **projection of v onto u** is the vector $\operatorname{proj}_{\boldsymbol{u}}(\boldsymbol{v})$ defined by

$$\operatorname{proj}_{\boldsymbol{u}}(\boldsymbol{v}) = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right) \boldsymbol{u}$$

- Parallel to $\boldsymbol{u}$
- What if the angle if obtuse?
- What if $\boldsymbol{u}$ is a unit vector?

Cartesian Coordinate System

- Standard unit vectors

$$\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n \in \mathbb{R}^n$$

- Any (point) vector $\boldsymbol{v} \in \mathbb{R}^n$ can be represented as a **linear combination** of the standard unit vectors:

$$\boldsymbol{v} = \sum_{j=1}^{n} (\boldsymbol{v} \cdot \boldsymbol{e}_j)\boldsymbol{e}_j = (\boldsymbol{v} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + \cdots + (\boldsymbol{v} \cdot \boldsymbol{e}_n)\boldsymbol{e}_n.$$

  Example:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The Cartesian coordinate of a (point) vector $\boldsymbol{v}$ is

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v} \cdot \boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{v} \cdot \boldsymbol{e}_n \end{bmatrix}.$$

Outline

Line Equations

- $y = mx + k$ (lines on the plane)
  - m: slope
  - k: y-intercept
- $ax + by = c$ (lines on the plane)
- $\boldsymbol{n} \cdot \boldsymbol{x} = 0$ (**normal form** of lines passing through the origin)
  - $\boldsymbol{x}$: position vectors on the line
  - $\boldsymbol{n}$: **normal vector**
- $\boldsymbol{x} = t\boldsymbol{d}$ (**vector form** of lines passing through the origin)
  - $\boldsymbol{d}$: direction vector

Line Equations (cont'd)

Definition

The **normal form of the equation of a line** ℓ in $\mathbb{R}^2$ is

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{p}) = 0 \text{ or } \boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$$

where $\boldsymbol{p}$ is a specific point on ℓ and $\boldsymbol{n} \neq \mathbf{0}$ is a normal vector for ℓ.

The **general form of the equation of** ℓ is

$$\boldsymbol{n} \cdot \boldsymbol{x} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = ax + by = c$$

where $\boldsymbol{n}$ is a normal vector for ℓ.

- What is the geometric meaning of c in the equation $\boldsymbol{n} \cdot \boldsymbol{x} = c$ when $\|\boldsymbol{n}\| = 1$?

Line Equations (cont'd)

Definition

The **vector form of the equation of a line** ℓ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{d}$$

where $\boldsymbol{p}$ is a specific point on ℓ and $\boldsymbol{d} \neq \mathbf{0}$ is a direction vector for ℓ.
The equation corresponding to the components of the vector form of the equation are called **parametric equations** of ℓ.

- What is the parametric equation of the line passing through two points P and Q? Let $\boldsymbol{p} = \overrightarrow{OP}$ and $\boldsymbol{q} = \overrightarrow{OQ}$.

Plane Equations

Definition

The **normal form of the equation of a plane** $\mathcal{P}$ in $\mathbb{R}^3$ is

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{p}) = 0 \text{ or } \boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$$

where $\boldsymbol{p}$ is a specific point on $\mathcal{P}$ and $\boldsymbol{n} \neq \mathbf{0}$ is a normal vector for $\mathcal{P}$.

The **general form of the equation of** $\mathcal{P}$ is

$$\boldsymbol{n} \cdot \boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz = d$$

where $\boldsymbol{n}$ is a normal vector for $\mathcal{P}$.

- Hyperplanes

Plane Equations (cont'd)

Definition

The **vector form of the equation of a plane** $\mathscr{P}$ in $\mathbb{R}^3$ is

$$\boldsymbol{x} = \boldsymbol{p} + s\boldsymbol{u} + t\boldsymbol{v}$$

where $\boldsymbol{p}$ is a point on $\mathscr{P}$ and $\boldsymbol{u}$ and $\boldsymbol{v}$ are direction vectors for $\mathscr{P}$ ($\boldsymbol{u}$ and $\boldsymbol{v}$ are non-zero and parallel to $\mathscr{P}$, but not parallel to each other).
The equations corresponding to the components of the vector form of the equation are called **parametric equations** of $\mathscr{P}$.

- What is the parametric equation of the plane passing through three points P, Q and R? Let $\boldsymbol{p} = \overrightarrow{OP}$, $\boldsymbol{q} = \overrightarrow{OQ}$ and $\boldsymbol{r} = \overrightarrow{OR}$.

Distance between a Point and a Hyperplane

Distance between a point and a line in 2D

- General form of line equation of ℓ: $ax + by = c$
- Point $B = (x_0, y_0)$

$$d(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$

Distance between a point and a plane in 3D

- General form of plane equation of $\mathcal{P}$: $ax + by + cz = d$
- Point $B = (x_0, y_0, z_0)$

$$d(B, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cross Product

Definition

The **cross product** of

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

is the vector $\boldsymbol{u} \times \boldsymbol{v}$ defined by

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

- $\|\boldsymbol{u} \times \boldsymbol{v}\| =?$
- What is the geometric meaning of $\|\boldsymbol{u} \times \boldsymbol{v}\|$?

Scalar Triple Product

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$. Then **the scalar triple product** of the three vectors is defined as

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$$

- $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$
- $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = -\boldsymbol{w} \cdot (\boldsymbol{v} \times \boldsymbol{u})$
- Geometric meaning: (signed) volume of the parallelepiped defined by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

Outline

Skipped.