Homework #6

June 13, 2011

1. If a subspace S_1 is contained in a subspace S_2 (i.e., $S_1 \subset S_2$), prove that $S_1^{\perp}$ contains $S_2^{\perp}$ (i.e., $S_2^{\perp} \subset S_1^{\perp}$). Hint: Prove that every vector $\boldsymbol{x} \in S_2^{\perp}$ is also contained in $S_1^{\perp}$ by showing that $\boldsymbol{x}$ is orthogonal to all the vectors in S_1.

2. Suppose an $n \times n$ matrix A is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the subspace spanned by which rows of A?

3. Let $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$.

   (a) Compute the two projection matrices P_1 and P_2 onto the lines through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively.

   (b) Compute $P_1 P_2$ and $P_2 P_1$.

   (c) Explain the result of (b) from the "transformation" point of view.

4. Let P be a projection matrix.

   (a) Show that $(I - P)^2 = I - P$.

   (b) Let P projects onto the column space of A. Then onto which fundamental subspace of A does $I - P$ project?

5. If an $m \times m$ matrix A satisfies $A^2 = A$ and $\text{rank}(A) = m$, $A = I$. Prove it.

6. Let $P^T = P$ and $P^2 = P$. If $\boldsymbol{p}_i$ is the i-th column of P, show that $\|\boldsymbol{p}_i\|^2$ is the same as the (i, i) element of P.

7. Let the matrix A have three columns and they are $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$. And let $\|\boldsymbol{a}_1\| = 1$, $\|\boldsymbol{a}_2\| = 2$, and $\|\boldsymbol{a}_3\| = 3$. Then what is $A^T A$?