# Homework #5

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1. A $3 \times 3$ matrix $B$ is known to have eigenvalues 0, 1, 2.
   (a) Find $\operatorname{rank}(B)$
   (b) Find $\det(B^T B)$

**Solution:**

(a) Since 0 has its algebraic multiplicity 1, its geometric multiplicity should be 1. Therefore,

$$\dim E_0 = \dim(\operatorname{null}(B - 0I)) = \dim \operatorname{null}(B) = \operatorname{nullity}(B) = 1.$$

Therefore,

$$\operatorname{rank}(B) = 3 - \operatorname{nullity}(B) = 2.$$

(b) Note that $\det B = 0 \cdot 1 \cdot 2 = 0$. Then,

$$\det(B^T B) = \det B^T \det B = (\det B)^2 = 0$$

2. Let

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$$

   (a) Without using Gaussian elimination, find $\operatorname{rank}(A)$.
   Hint: For any $\boldsymbol{x}$, how does $A\boldsymbol{x}$ look like?
   (b) Without using Gaussian elimination, find the eigenvalues and eigenspaces.
   Hint:
   - Try to find $\boldsymbol{x}$ such that

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \boldsymbol{x} = \lambda \boldsymbol{x}.$$

   - What is $\operatorname{nullity}(A)$? How can we find the vectors in $\operatorname{null}(A)$ easily?

**Solution:**

(a) For any $\boldsymbol{y}$,

$$A\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \left( \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \boldsymbol{y} \right) = \left( \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \boldsymbol{y} \right) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore $A\boldsymbol{y}$ is a multiple of $(1, 2, 1)$ hence

$$\text{col}(A) = \text{span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right)$$

and $\text{rank}(A) = 1$.

(b)

$$A\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}\left(\begin{bmatrix}2 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = 6\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Therefore

$$E_6 = \text{span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right).$$

On the other hands, if we choose any vector $\boldsymbol{x}$ orthogonal to $(2, 1, 2)$,

$$A\boldsymbol{x} = \begin{bmatrix}1\\2\\1\end{bmatrix}\left(\begin{bmatrix}2 & 1 & 2\end{bmatrix}\boldsymbol{x}\right) = \mathbf{0}.$$

hence $\boldsymbol{x} \in E_0 = \text{null}(A - 0I) = \text{null}(A)$. From (a), $\text{nullity}(A) = 3 - \text{rank}(A) = 2$ therefore we have two linearly independent such vectors, e.g., $(1, -2, 0)$ and $(0, -2, 1)$. Therefore

$$E_0 = \text{span}\left(\begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\1\end{bmatrix}\right).$$

3. Let $a + b = c + d$.

$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}.$$

(a) Show that $(1, 1)$ is an eigenvector of $A$.

(b) Find both eigenvalues of $A$.

**Solution:**

(a)

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a+b\\c+d\end{bmatrix} = (a+b)\begin{bmatrix}1\\1\end{bmatrix}.$$

Therefore, $(1, 1)$ is the eigenvector with its eigenvalue $a + b$.

(b) Let $\lambda_1 = a + b$ and $\lambda_2$ are two eigenvalues of $A$

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

~~Therefore $\lambda_1 + \lambda_2 = a + d + \lambda_2 = a + d$ hence $\lambda_2 = 0$.~~ Therefore $\lambda_1 + \lambda_2 = a + b + \lambda_2 = a + d$ hence $\lambda_2 = d - b$.

4. Suppose $A$ has eigenvalues 0, 3, 5 with linearly independent eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.

   (a) Give a basis for null($A$) and a basis for col($A$).
   Hint:

   - null($A$) $= E_0$.
   - Consider the linear combination $c_1\boldsymbol{v} + c_2\boldsymbol{w}$.

   (b) Show that $A\boldsymbol{x} = \boldsymbol{u}$ has no solution.
   Hint: If it did, then ( ) would be in col($A$) and this contradicts the assumption.

**Solution:**

(a)

$$\text{null}(A) = \text{null}(A - 0I) = E_0 = \text{span}\,(\boldsymbol{u})\,.$$

For any linear combination $c_1\boldsymbol{v} + c_2\boldsymbol{w}$,

$$c_1\boldsymbol{v} + c_2\boldsymbol{w} = \frac{c_1}{3}A\boldsymbol{v} + \frac{c_2}{5}A\boldsymbol{w} = A\left(\frac{c_1}{3}\boldsymbol{v} + \frac{c_2}{5}\boldsymbol{w}\right) \in \text{col}(A),$$

therefore

$$\text{col}(A) = \text{span}\,(\boldsymbol{v}, \boldsymbol{w})\,.$$

(b)

$$A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w} = \frac{1}{3}A\boldsymbol{v} + \frac{1}{5}A\boldsymbol{w} = A\left(\frac{\boldsymbol{v}}{3} + \frac{\boldsymbol{w}}{5}\right)$$

All solutions are of the form

$$\frac{\boldsymbol{v}}{3} + \frac{\boldsymbol{w}}{5} + c\boldsymbol{u}.$$

(c) Assume that $A\boldsymbol{x} = \boldsymbol{u}$ has a solution $\boldsymbol{x}_0$. Then $\boldsymbol{u} \in \text{col}(A)$, but $\boldsymbol{u}$ is linearly independent of both $\boldsymbol{v}$ and $\boldsymbol{w}$ therefore cannot be in col($A$).

5. If $A$ has an eigenvalue $\lambda_1 = 2$ with its eigenvector $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is $A$?

**Solution:**

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

6. Let the $n \times n$ matrix $A$ have the eigenvalues $\lambda_1, \ldots, \lambda_n$ and be diagonalizable. Find the eigenvalues of the $2n \times 2n$ block matrix

$$B = \begin{bmatrix} A & O \\ O & 2A \end{bmatrix}.$$

**Solution:**

$$B = \begin{bmatrix} A & O \\ O & 2A \end{bmatrix} = \begin{bmatrix} PDP^{-1} & O \\ O & 2PDP^{-1} \end{bmatrix} = \begin{bmatrix} P & O \\ O & P \end{bmatrix} \begin{bmatrix} D & O \\ O & 2D \end{bmatrix} \begin{bmatrix} P^{-1} & O \\ O & P^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} P & O \\ O & P \end{bmatrix} \begin{bmatrix} D & O \\ O & 2D \end{bmatrix} \begin{bmatrix} P & O \\ O & P \end{bmatrix}^{-1}$$

7. For an $n \times n$ matrix $A$, suppose $A^2 = A$.
   (a) Show that 0 is an eigenvalue of $A$ and $E_0 = \text{null}(A)$.
   (b) Show that 1 is an eigenvalue of $A$ and $E_1 = \text{col}(A)$.
   (c) Show that $A$ is diagonalizable.
   Hint: $A$ is diagonalizable if the sum of all the dimensions of eigenspaces (geometric multiplicities) is $n$.

**Solution:**

(a) There is an error in this question. The question should be "For $A \neq I$, show that 0 is an eigenvalue of $A$ and $E_0 = \text{null}(A)$.

$$A^2 = A \rightarrow A(A - I) = O$$

Let $B := A - I$. Since $A \neq I$, there is at least one column of $B$ which is not a zero vector. Let $\boldsymbol{x}$ is that vector. Then, if we consider only that column in $AB = O$, $A\boldsymbol{x} = \mathbf{0}$ therefore $\boldsymbol{x}$ is an eigenvector of $A$ corresponding to the eigenvalue 0.

For any $\boldsymbol{x} \in E_0$, $A\boldsymbol{x} = \mathbf{0}$ therefore $\boldsymbol{x} \in \text{null}(A)$. Also, for any $\boldsymbol{y} \in \text{null}(A)$, $A\boldsymbol{y} = \mathbf{0} = 0\boldsymbol{y}$ therefore $\boldsymbol{y} \in E_0$. Overall, $E_0 = \text{null}(A)$.

(b) Let
$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}.$$
Then, from $A^2 = A$,
$$A \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}.$$
Therefore for any $\boldsymbol{a}_i$,
$$A\boldsymbol{a}_i = \boldsymbol{a}_i$$
hence 1 is an eigenvalue of $A$. And
$$E_1 = \text{span}\left(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\right) = \text{col}(A).$$

(c) Since $\text{rank}(A) = \dim \text{col}(A) = \dim E_1$ and $\text{nullity}(A) = \dim \text{null}(A) = \dim E_0$, by the rank theorem,
$$\text{rank}(A) + \text{nullity}(A) = \dim E_1 + \dim E_0 = n.$$
Therefore, the union of the bases of $E_0$ and $E_1$ has $n$ linearly independent vector, which means that $A$ is diagonalizable. (Theorem 4.27 on p.304)

8. Suppose that both $A$ and $B$ are diagonalizable by the same $P$:
$$A = PD_1P^{-1} \quad \text{and} \quad B = PD_2P^{-1}.$$

Show that $AB = BA$.

**Solution:**

$$AB = PD_1P^{-1}PD_2P^{-1} = PD_1D_2P^{-1}$$

and

$$BA = PD_2P^{-1}PD_1P^{-1} = PD_2D_1P^{-1}$$

but

$$D_1D_2 = D_2D_1$$

since $D_1$ and $D_2$ are diagonal.

9. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $AB = BA$.

(a) Show that $B$ is diagonal.

(b) Show that $A$ and $B$ have the same eigenvectors.

**Solution:**

(a)

$$AB = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix}$$

Since $AB = BA$, $b = 2b$ and $c = 2c$ hence $b = c = 0$ and $B$ is diagonal.

(b) Since both $A$ and $B$ are diagonal, they are diagonalized by the same $I$, but with different eigenvalues.