

Linear Algebra

Chapter 4: Eigenvalues and Eigenvectors

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Dynamical System

- ▶ For $A \in \mathbb{R}^{n \times n}$, $\lim_{k \rightarrow \infty} A^k = ?$
 - ▶ Try the Octave demos your selves!
1. Complete graphs (K4.m)
 2. Petersen graph (Petersen.m)
 3. Cyclic graphs
 - 3.1 Odd number of nodes (C5.m)
 - 3.2 Even number of nodes (C6.m)
 4. Complete bipartite graphs (K3_3.m)
- ▶ Steady state vector

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Eigenvalue Problem

- ▶ For a matrix $A \in \mathbb{R}^{n \times n}$, which vectors keep (or reverse) their directions after transformed by A ? In other words, who satisfy $Ax = \lambda x$? \rightarrow “Eigenvalue problem”
- ▶ eigen- [áigən]: “own” or “characteristic of”

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there is a nonzero x such that $Ax = \lambda x$. Such a vector x is called an **eigenvector** of A corresponding to λ .

- ▶ Why are they important?
- ▶ Do they exist for *any* matrix?
- ▶ Is there only one eigenvector for an eigenvalue?
- ▶ Is there only one eigenvalue for an eigenvector?
- ▶ Given an eigenvalue, how can we find the corresponding eigenvectors? \rightarrow Example 4.2
- ▶ How can we find eigenvalues?

Eigenspace

► Example 4.2

→ “The set of all eigenvectors corresponding to an eigenvalue λ of an $n \times n$ matrix A is just the set of *nonzero* vectors in the null space of $A - \lambda I$.”

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . The collection of all eigenvectors corresponding to λ , together with the zero vector, is called the **eigenspace** of λ and is denoted by E_λ .

Geometric Interpretation of Eigenvectors

- ▶ Ax and λx are parallel, i.e., x is an eigenvector of A iff A transforms x into a parallel vector.
- ▶ Examples: Scaling, reflection (Ex 4.4), rotation
- ▶ Only the direction of an eigenvector matters.
→ Only unit vectors need to be considered.

Finding Eigenvalues

- ▶ For $A \in \mathbb{R}^{n \times n}$, the eigenvectors of λ are the nonzero vectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$.
 - $A\mathbf{x} - \lambda I\mathbf{x} = \mathbf{0} \rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$
 - The eigenspace E_λ is the non-trivial null space of $A - \lambda I$.
- ▶ For 2×2 matrices,
 - a matrix has a non-trivial null space iff it is non-invertible.
 - a matrix is non-invertible iff its determinant is zero.
 - λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$ (Example 4.5)
- ▶ Can be generalized to any square matrices.
 - What are the determinants for $n \times n$ matrices?

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Determinants

▶ Notation: $\det A = |A|$

▶ 1×1 matrices

$$\det A = |a| = a \quad (\text{Not the absolute value})$$

▶ 2×2 matrices

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

▶ 3×3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ?$$

Determinant of a 3×3 Matrix

Definition

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the **determinant** of A is the scalar

$$\det A = |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

▶ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

▶ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

▶ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Determinant of a 3×3 Matrix (cont'd)

- ▶ With A_{ij} defined as the submatrix of A obtained by deleting row i and column j ,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} \\ &= \sum_{j=1}^3 (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

- ▶ $\det A_{ij}$ is called the (i, j) -**minor** of A .
- ▶ Computed with respect to the first row.
 - Why row not column? Why the first row?
 - Can be generalized to columns or other rows (The Laplace Expansion Theorem)
- ▶ Another method (See (2) on p.264)

$$\begin{aligned}|A| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}\end{aligned}$$

Determinants of $n \times n$ Matrices

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geq 2$. Then the **determinant** of A is the scalar

$$\begin{aligned}\det A = |A| &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

► **Cofactor expansion along the first row:**

With (i, j) -**cofactor** of A defined as

$$C_{ij} = (-1)^{i+j} \det A_{ij},$$

the definition becomes

$$\det A = \sum_{j=1}^n a_{1j} C_{1j}.$$

The Laplace Expansion Theorem

Theorem 4.1: The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^n a_{ij}C_{ij}$$

(which is the **cofactor expansion along the i th row**) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^n a_{ij}C_{ij}$$

(the **cofactor expansion along the j th column**)

- ▶ Recursive computation \rightarrow Not efficient (Note on p.268)
- ▶ Most useful when the matrix contains a row or column with lots of zeros. Why? (Example 4.11)

Determinants of Triangular Matrices

- ▶ The Laplace expansion theorem is particularly useful when the matrix is (upper or lower) triangular.

Theorem 4.2

The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix then

$$\det A = a_{11}a_{22} \cdots a_{nn}$$

- ▶ Why? (Example 4.12)

Properties of Determinants

Theorem 4.3

Let $A = [a_{ij}]$ be a square matrix.

- a. If A has a zero row (column), then $\det A = 0$.
- b. If B is obtained by interchanging two rows (columns) of A , then $\det B = -\det A$.
- c. If A has two identical rows (columns), then $\det A = 0$.
- d. If B is obtained by multiplying a row (column) of A by k , then $\det B = k \det A$.
- e. If A , B , and C are identical except that the i th row (column) of C is the sum of the i th rows (columns) of A and B , then $\det C = \det A + \det B$.
- f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

Determinants of Elementary Matrices

- ▶ (b), (d) and (f) of the properties are related to elementary row operations.
- ▶ What are the determinants of elementary matrices?

Theorem 4.4

Let E be an $n \times n$ elementary matrix.

- If E results from interchanging two rows of I_n , then $\det E = -1$.
 - If E results from multiplying one row of I_n by k , then $\det E = k$.
 - If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.
- ▶ Determinants of all the elementary matrices are nonzero.

Determinants of Elementary Matrices (cont'd)

- ▶ If $B = EA$, $\det B = ?$

Lemma 4.5

Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix.
Then

$$\det(EB) = (\det E)(\det B)$$

- ▶ How about $\det(AB)$ when A is NOT an elementary matrix?
→ Theorem 4.8 (p.272)

Theorem 4.6

A square matrix A is invertible if and only if $\det A \neq 0$.

Determinants and Matrix Operations

How can we write the followings in terms of $\det A$ and $\det B$?

- ▶ $\det(kA) = ?$
- ▶ $\det(A + B) = ?$
- ▶ $\det(AB) = ?$
- ▶ $\det(A^{-1}) = ?$
- ▶ $\det(A^T) = ?$

$\det(kA)$ and $\det(A + B)$

- ▶ $\det(kA)$

Theorem 4.7

If A is an $n \times n$ matrix, then

$$\det(kA) = k^n \det A$$

- ▶ See Theorem 4.3(d).
- ▶ $\det(A + B)$
 - ▶ $\det(A + B) = \det A + \det B$? → No!
 - ▶ No general formula.

$\det(AB)$, $\det(A^{-1})$ and $\det(A^T)$

- ▶ $\det(AB)$

Theorem 4.8

If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

- ▶ $\det(A^{-1})$

Theorem 4.9

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

- ▶ $\det(A^T)$

Theorem 4.10

For any square matrix A ,

$$\det A = \det A^T$$

Cramer's Rule

- ▶ What is the relation between the solution of a linear system and its determinant?

Theorem 4.11: Cramer's Rule

Let A be an invertible $n \times n$ matrix and let \mathbf{b} be a vector in \mathbb{R}^n . Then the unique solution \mathbf{x} of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det A} \quad \text{for } i = 1, \dots, n$$

- ▶ $A_i(\mathbf{b})$: the matrix obtained by replacing the i th column of A by \mathbf{b}

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \cdots \quad \mathbf{b} \quad \cdots \quad \mathbf{a}_n]$$

- ▶ Requires to compute determinants
→ Computationally inefficient except small systems.

Adjoint

- ▶ What is the formula of the inverse of a matrix in terms of determinants?
- ▶ What is the solution of the equation $AX = I$?

$$A\mathbf{x}_1 = \mathbf{e}_1 \quad A\mathbf{x}_2 = \mathbf{e}_2 \quad \cdots \quad A\mathbf{x}_n = \mathbf{e}_n$$

Theorem 4.12

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

- ▶ $\operatorname{adj} A := [C_{ji}] = [C_{ij}]^T$: the **adjoint** (or **adjugate**) of A

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Finding Eigenvalues

- ▶ How to compute eigenvalues of a matrix?
- ▶ How many eigenvalues does a matrix have?
- ▶ λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.

The eigenvalues of a square matrix A are precisely the solutions λ of the equation

$$\det(A - \lambda I) = 0$$

- ▶ What does $\det(A - \lambda I)$ look like?
 - Polynomial in λ of degree n
 - (Characteristic polynomial of A)**
 - At most n distinct eigenvalues

Finding Eigenvalues and Eigenvectors

Procedure

Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$ of A .
 2. Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ .
 3. For each eigenvalue λ , find the null space of the matrix $A - \lambda I$. This is the eigenspace E_λ , the nonzero vectors of which are the eigenvectors of A corresponding to λ .
 4. Find a basis for each eigenspace.
- ▶ **Algebraic multiplicity** of an eigenvalue: multiplicity as a root of the characteristic equation.
 - ▶ **Geometric multiplicity** of an eigenvalue λ : $\dim E_\lambda$
 - ▶ Geometric multiplicity \leq Algebraic multiplicity
(Lemma 4.26 on p.303)

Eigenvalues of Triangular Matrices

Theorem 4.15

The eigenvalues of a triangular matrix are the entries on its main diagonal.

- ▶ Why? (See Theorem 4.2 on p.268)

Fundamental Theorem of Invertible Matrices: Ver. 3

Theorem 3.27

Let A be an $n \times n$ matrix. The following statements are equivalent:

- a. A is invertible.
- b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every \mathbf{b} in \mathbb{R}^n .
- c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- d. The reduced row echelon form of A is I_n .
- e. A is a product of elementary matrices.
- f. $\text{rank}(A) = n$
- g. $\text{nullity}(A) = 0$
- h. The column vectors of A are linearly independent.
- i. The column vectors of A span \mathbb{R}^n .
- j. The column vectors of A form a basis for \mathbb{R}^n .
- k. The row vectors of A are linearly independent.
- l. The row vectors of A span \mathbb{R}^n .
- m. The row vectors of A form a basis for \mathbb{R}^n .
- n. $\det A \neq 0$
- o. 0 is not an eigenvalue of A .

Eigenvalue of A^k and A^{-1}

Theorem 4.18

Let A be a square matrix with eigenvalue λ and corresponding eigenvector \mathbf{x} .

- For any positive integer k , λ^k is an eigenvalue of A^k with corresponding eigenvector \mathbf{x} .
- If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector \mathbf{x} .
- For any integer k , λ^k is an eigenvalue of A^k with corresponding eigenvector \mathbf{x} .

Computing $A^k \mathbf{x}$

- ▶ How to compute $A^k \mathbf{x}$, where \mathbf{x} is not an eigenvector of A ? (Example 4.21)

Theorem 4.19

Suppose the $n \times n$ matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. If \mathbf{x} is a vector in \mathbb{R}^n that can be expressed as a linear combination of these eigenvectors—say,

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_m \mathbf{v}_m$$

then, for any integer k ,

$$A^k \mathbf{x} = c_1 \lambda_1^k \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 + \dots + c_m \lambda_m^k \mathbf{v}_m$$

Theorem 4.20

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$. Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are linearly independent.

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Why Diagonalize Matrices?

- ▶ Triangular and diagonal matrices are good.
→ How can we relate a square matrix to a triangular or diagonal one keeping the eigenvalues?
- ▶ Gaussian elimination?
→ Eigenvalues are not preserved.
- ▶ Diagonalization

Similar Matrices

Definition

Let A and B be $n \times n$ matrices. We say that A is **similar to** B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B , we write $A \sim B$.

- ▶ Equivalent to “ $A = PBP^{-1}$ ” or “ $AP = PB$.”
- ▶ “ $A \sim B$ ” does not necessarily mean “ $B \sim A$.”
- ▶ P depends on A and B . Is it unique?

Theorem 4.21

Let A , B and C be $n \times n$ matrices.

- $A \sim A$. (Reflexivity)
 - If $A \sim B$, then $B \sim A$. (Symmetry)
 - If $A \sim B$ and $B \sim C$, then $A \sim C$. (Transitivity)
- ▶ **Equivalent relation**

Similar Matrices (cont'd)

Theorem 4.22

Let A and B be $n \times n$ matrices with $A \sim B$. Then

- $\det A = \det B$.
 - A is invertible if and only if B is invertible.
 - A and B have the same rank.
 - A and B have the same characteristic polynomial.
 - A and B have the same eigenvalues.
- The converse is not necessarily true. (See Remark on p.300.)

Diagonalization

- ▶ Good if a square matrix is similar to a diagonal matrix.
- ▶ Is it always possible?

Definition

An $n \times n$ matrix A is **diagonalizable** if there is a diagonal matrix D such that A is similar to D —that is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

Diagonalization (cont'd)

- ▶ How can we find D and P ?

Theorem 4.23

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are eigenvalues of A corresponding to the eigenvectors in P in the same order.

- ▶ Is a non-invertible matrix diagonalizable? (Example 4.26)

Diagonalization (cont'd)

- ▶ How can we check if the eigenvectors are linearly independent? (Example 4.26)

Theorem 4.24

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of A . If \mathcal{B}_i is a basis for the eigenspace E_{λ_i} , then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_k$ (i.e., the total collection of basis vectors for all the eigenspaces) is linearly independent.

Theorem 4.25

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

The Diagonalization Theorem

Lemma 4.26

If A is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

Theorem 4.27

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_k$. The following statements are equivalent:

- A is diagonalizable.
 - The union \mathcal{B} of the bases of the eigenspaces of A (as in Theorem 4.24) contains n vectors.
 - The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.
- Computing A^k (Example 4.29)
 $A^k = PD^kP^{-1}$ for all $k \geq 1$.

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