

Linear Algebra

Chapter 1: Vectors

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Review on vectors

What you learned in high school?
(Wikipedia korea, springnote.com)

- ▶ Definition
- ▶ Vector representation using coordinates
- ▶ Algebra of vectors
- ▶ Dot product (점곱) or scalar product (스칼라곱)
 - * Dot(scalar) product *is* a inner product (내적), but *not vice versa*.
- ▶ Length of a vector

→ Watch “Vectors” video on YouTube

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...Enjoy with your friends!

Outline

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Vectors

- ▶ Direction + magnitude
- ▶ **Displacement** from **initial point (tail)** to **terminal point (head)**
- ▶ Notation: \overrightarrow{AB} or v
- ▶ Position vectors: 1-to-1 correspondence between **points** on the plane and (2D) **vectors**
- ▶ **Points \neq vectors**
- ▶ **Representation** \rightarrow Why?
 - ▶ **Coordinate system** (e.g., Cartesian, polar)
 - ▶ Components
 - ▶ **Ordered** pair of real numbers
 - ▶ Notation:
Row vector $[x, y]$ or column vectors $v = \begin{bmatrix} x \\ y \end{bmatrix}$
cf) Notation for points: $P = (x, y)$
- ▶ Zero vector (**0**)
- ▶ (Real) vector space \mathbb{R}^2 : “the set of all vectors with two components”

Vector Algebra

Let $\mathbf{u} := [u_1, u_2]$ and $\mathbf{v} := [v_1, v_2]$.

($\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$)

- ▶ Addition: $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$
 - ▶ 'Head-to-tail rule' or 'parallelogram rule'
- ▶ Scalar multiplication: $c\mathbf{u} = c[v_1, v_2] = [cv_1, cv_2]$
 - ▶ **Scalar**: constant
- ▶ Subtraction: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = [u_1 - v_1, u_2 - v_2]$

Higher Dimensional Vectors

Vectors in \mathbb{R}^n

- ▶ **Ordered n -tuples** of real numbers

- ▶ $\mathbf{v} = [v_1, \dots, v_n] = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$

- ▶ Geometric method doesn't work anymore
→ We need analytic method using n -tuples.

Algebraic properties of vectors in \mathbb{R}^n

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

- a. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)
- b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)
- c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- e. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity)
- f. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity)
- g. $c(d\mathbf{u}) = (cd)\mathbf{u}$
- h. $1\mathbf{u} = \mathbf{u}$

→ Try to prove them yourselves!

Linear Combination

Definition

A vector \mathbf{v} is a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ if there are scalars c_1, c_2, \dots, c_k such that

$$\mathbf{v} = \sum_{j=1}^k c_j \mathbf{v}_j = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k.$$

The scalars c_1, c_2, \dots, c_k are called the **coefficients** of the linear combination.

- ▶ Do such scalars c_1, c_2, \dots, c_k always exist for any \mathbf{v} and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$?
- ▶ How many vectors do we need? I.e., how large should k be?
- ▶ Is it enough just to have sufficiently many vectors?
- ▶ How is the situation different between \mathbb{R} and \mathbb{R}^2 ?
- ▶ **Coordinate axes** and **coordinate grid**

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Dot Product

Can we multiply a vector with another vector?

→ **Dot (scalar) product** and **cross product**

Definition

If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

then the **dot product** $\mathbf{u} \cdot \mathbf{v}$ of \mathbf{u} and \mathbf{v} is defined by

$$\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^n u_j v_j = u_1 v_1 + \cdots + u_n v_n \in \mathbb{R}.$$

- ▶ The dot product is one of the more general **inner product**.
(Chap. 7)

Properties of Dot Product

Theorem 1.2

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

- a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutativity)
- b. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributivity)
- c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- d. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ iff (if and only if) $\mathbf{u} = \mathbf{0}$.

→ Try to prove them yourselves!

▶ $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = ?$

Length

Definition

The **length** (or **norm**) of a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$$

Theorem 1.3

Let $\mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

- $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = \mathbf{0}$
- $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

Unit Vectors

- ▶ Vectors with length 1
- ▶ **Normalization:** to make the length of a vector 1
 $v \rightarrow (1/\|v\|)v$
- ▶ **Standard unit vectors**

Examples:

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

More on Length

Theorem 1.4: The Cauchy-Schwarz inequality

For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

Theorem 1.5: The triangle inequality

For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Distance

Definition

The **distance** $d(\mathbf{u}, \mathbf{v})$ between (position) vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Angles

Definition: Angle

For nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

Definition: Orthogonality

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** to each other ($\mathbf{u} \perp \mathbf{v}$) if $\mathbf{u} \cdot \mathbf{v} = 0$.

- ▶ Zero vector

Theorem 1.6: Pythagoras' Theorem

For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

iff $\mathbf{u} \perp \mathbf{v}$.

Projections

Definition

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$, the the **projection of \mathbf{v} onto \mathbf{u}** is the vector $\text{proj}_{\mathbf{u}}(\mathbf{v})$ defined by

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}$$

- ▶ Parallel to \mathbf{u}
- ▶ What if the angle is obtuse?
- ▶ What if \mathbf{u} is a unit vector?

Cartesian Coordinate System

- ▶ Standard unit vectors

$$\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{R}^n$$

- ▶ Any (point) vector $\mathbf{v} \in \mathbb{R}^n$ can be represented as a **linear combination** of the standard unit vectors:

$$\mathbf{v} = \sum_{j=1}^n (\mathbf{v} \cdot \mathbf{e}_j) \mathbf{e}_j = (\mathbf{v} \cdot \mathbf{e}_1) \mathbf{e}_1 + \dots + (\mathbf{v} \cdot \mathbf{e}_n) \mathbf{e}_n.$$

Example:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- ▶ The Cartesian coordinate of a (point) vector \mathbf{v} is

$$\mathbf{v} = \begin{bmatrix} \mathbf{v} \cdot \mathbf{e}_1 \\ \vdots \\ \mathbf{v} \cdot \mathbf{e}_n \end{bmatrix}.$$

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Line Equations

- ▶ $y = mx + k$ (lines on the plane)
 - ▶ m : slope
 - ▶ k : y -intercept
- ▶ $ax + by = c$ (lines on the plane)
- ▶ $\mathbf{n} \cdot \mathbf{x} = 0$ (**normal form** of lines passing through the origin)
 - ▶ \mathbf{x} : position vectors on the line
 - ▶ \mathbf{n} : **normal vector**
- ▶ $\mathbf{x} = t\mathbf{d}$ (**vector form** of lines passing through the origin)
 - ▶ \mathbf{d} : direction vector

Line Equations (cont'd)

Definition

The **normal form of the equation of a line** ℓ in \mathbb{R}^2 is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \text{ or } \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where \mathbf{p} is a specific point on ℓ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for ℓ .

The **general form of the equation of ℓ** is

$$\mathbf{n} \cdot \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = ax + by = c$$

where \mathbf{n} is a normal vector for ℓ .

- ▶ What is the geometric meaning of c in the equation $\mathbf{n} \cdot \mathbf{x} = c$ when $\|\mathbf{n}\| = 1$?

Line Equations (cont'd)

Definition

The **vector form of the equation of a line** ℓ in \mathbb{R}^2 or \mathbb{R}^3 is

$$\mathbf{x} = \mathbf{p} + t\mathbf{d}$$

where \mathbf{p} is a specific point on ℓ and $\mathbf{d} \neq \mathbf{0}$ is a direction vector for ℓ .

The equation corresponding to the components of the vector form of the equation are called **parametric equations** of ℓ .

- ▶ What is the parametric equation of the line passing through two points P and Q ? Let $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$.

Plane Equations

Definition

The **normal form of the equation of a plane** \mathcal{P} in \mathbb{R}^3 is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \text{ or } \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where \mathbf{p} is a specific point on \mathcal{P} and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for \mathcal{P} .

The **general form of the equation of \mathcal{P}** is

$$\mathbf{n} \cdot \mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz = d$$

where \mathbf{n} is a normal vector for \mathcal{P} .

- ▶ Hyperplanes

Plane Equations (cont'd)

Definition

The **vector form of the equation of a plane** \mathcal{P} in \mathbb{R}^3 is

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$

where \mathbf{p} is a point on \mathcal{P} and \mathbf{u} and \mathbf{v} are direction vectors for \mathcal{P} (\mathbf{u} and \mathbf{v} are non-zero and parallel to \mathcal{P} , but not parallel to each other). The equations corresponding to the components of the vector form of the equation are called **parametric equations** of \mathcal{P} .

- ▶ What is the parametric equation of the plane passing through three points P , Q and R ? Let $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$ and $\mathbf{r} = \overrightarrow{OR}$.

Distance between a Point and a Hyperplane

Distance between a point and a line in 2D

- ▶ General form of line equation of ℓ : $ax + by = c$
- ▶ Point $B = (x_0, y_0)$

$$d(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$

Distance between a point and a plane in 3D

- ▶ General form of plane equation of \mathcal{P} : $ax + by + cz = d$
- ▶ Point $B = (x_0, y_0, z_0)$

$$d(B, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cross Product

Definition

The **cross product** of

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

is the vector $\mathbf{u} \times \mathbf{v}$ defined by

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

- ▶ $\|\mathbf{u} \times \mathbf{v}\| = ?$
- ▶ What is the geometric meaning of $\|\mathbf{u} \times \mathbf{v}\|$?

Scalar Triple Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then **the scalar triple product** of the three vectors is defined as

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

- ▶ $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$
- ▶ $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$
- ▶ Geometric meaning: (signed) volume of the parallelepiped defined by \mathbf{u} , \mathbf{v} and \mathbf{w} .

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