

Homework #5 Solution

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Excercise 5.1

33 (a) $A(A^T + B^T)B = AA^TB + AB^TB = (AA^T)B + A(B^TB) = B + A = A + B.$

(b) By (a),

$$\det(A + B) = \det(A(A^T + B^T)B) = \det A \det(A^T + B^T) \det B.$$

Since $(A + B)^T = A^T + B^T$ and by Theorem 4.10 (p.273),

$$\det(A^T + B^T) = \det(A + B).$$

Therefore,

$$\begin{aligned} \det(A + B) - \det(A(A^T + B^T)B) &= \det(A + B) - \det A \det(A^T + B^T) \det B \\ &= (1 - \det A \det B) \det(A + B) \\ &= (1 + (\det A)^2) \det(A + B) \\ &\quad \text{(since } \det A + \det B = 0\text{)} \\ &= 0. \end{aligned}$$

Since $1 + (\det A)^2 \neq 0$, $\det(A + B) = 0$ and therefore $A + B$ is not invertible.

34 Note that $Q = Q^T$ since

$$\left[\begin{array}{c|c} x_1 & \mathbf{y}^T \\ \hline \mathbf{y} & I - \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T \end{array} \right]^T = \left[\begin{array}{c|c} x_1 & \mathbf{y}^T \\ \hline (\mathbf{y}^T)^T & \left(I - \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T\right)^T \end{array} \right] = \left[\begin{array}{c|c} x_1 & \mathbf{y}^T \\ \hline \mathbf{y} & I - \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T \end{array} \right].$$

$$\begin{aligned} QQ^T = Q^2 &= \left[\begin{array}{c|c} x_1 & \mathbf{y}^T \\ \hline \mathbf{y} & I - \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T \end{array} \right]^2 \\ &= \left[\begin{array}{c|c} x_1^2 + \mathbf{y}^T \mathbf{y} & x_1 \mathbf{y}^T + \mathbf{y}^T \left(I - \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T\right) \\ \hline \mathbf{y} x_1 + \left(I - \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T\right) \mathbf{y} & \mathbf{y}\mathbf{y}^T + \left(I - \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T\right)^2 \end{array} \right]^2 \\ &= \left[\begin{array}{c|c} x_1^2 + \mathbf{y} \cdot \mathbf{y} & x_1 \mathbf{y}^T + \mathbf{y}^T - \left(\frac{1}{1-x_1}\right) \mathbf{y}^T \mathbf{y}\mathbf{y}^T \\ \hline x_1 \mathbf{y} + \mathbf{y} - \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T \mathbf{y} & \mathbf{y}\mathbf{y}^T + I - 2 \left(\frac{1}{1-x_1}\right) \mathbf{y}\mathbf{y}^T + \left(\frac{1}{1-x_1}\right)^2 \mathbf{y}\mathbf{y}^T \mathbf{y}\mathbf{y}^T \end{array} \right] \end{aligned}$$

(a) $x_1^2 + \mathbf{y} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{x} = 1.$

(b)

$$\begin{aligned} x_1 \mathbf{y}^T + \mathbf{y}^T - \left(\frac{1}{1-x_1} \right) \mathbf{y}^T \mathbf{y} \mathbf{y}^T &= (x_1 + 1) \mathbf{y}^T - \left(\frac{1}{1-x_1} \right) (\mathbf{y} \cdot \mathbf{y}) \mathbf{y}^T \\ &= (x_1 + 1) \mathbf{y}^T - \left(\frac{1}{1-x_1} \right) (1-x_1^2) \mathbf{y}^T \\ &= (x_1 + 1) \mathbf{y}^T - (1+x_1^2) \mathbf{y}^T \\ &= \mathbf{0}^T. \end{aligned}$$

(c)

$$\begin{aligned} x_1 \mathbf{y} + \mathbf{y} - \left(\frac{1}{1-x_1} \right) \mathbf{y} \mathbf{y}^T \mathbf{y} &= (x_1 + 1) \mathbf{y} + - \left(\frac{1}{1-x_1} \right) \mathbf{y} (\mathbf{y} \cdot \mathbf{y}) \\ &= (x_1 + 1) \mathbf{y} + - \left(\frac{1}{1-x_1} \right) \mathbf{y} (1-x_1^2) \\ &= (x_1 + 1) \mathbf{y} + -(1+x_1) \mathbf{y} = \mathbf{0}. \end{aligned}$$

(d)

$$\begin{aligned} \mathbf{y} \mathbf{y}^T + I - 2 \left(\frac{1}{1-x_1} \right) \mathbf{y} \mathbf{y}^T + \left(\frac{1}{1-x_1} \right)^2 \mathbf{y} \mathbf{y}^T \mathbf{y} \mathbf{y}^T \\ &= I + \left(1 - 2 \left(\frac{1}{1-x_1} \right) \right) \mathbf{y} \mathbf{y}^T + \left(\frac{1}{1-x_1} \right)^2 \mathbf{y} (\mathbf{y} \cdot \mathbf{y}) \mathbf{y}^T \\ &= I - \left(\frac{1+x_1}{1-x_1} \right) \mathbf{y} \mathbf{y}^T + \left(\frac{1-x_1^2}{(1-x_1)^2} \right) \mathbf{y} \mathbf{y}^T \\ &= I - \left(\frac{1+x_1}{1-x_1} \right) \mathbf{y} \mathbf{y}^T + \left(\frac{1+x_1}{1-x_1} \right) \mathbf{y} \mathbf{y}^T \\ &= I. \end{aligned}$$

Therefore, $QQ^T = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & I_{n-1} \end{bmatrix} = I_n$ and hence Q is orthogonal.

Excercise 5.2

14 Let a vector in W^\perp be $\mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. Since \mathbf{x} should be orthogonal to \mathbf{w}_1 ,

\mathbf{w}_2 and \mathbf{w}_3 , we get

$$\begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \mathbf{w}_3^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 & 6 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 & -3 \\ 2 & 2 & 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Gaussian elimination,

$$\begin{aligned} \begin{bmatrix} 4 & 6 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 & -3 \\ 2 & 2 & 2 & -1 & 2 \end{bmatrix} &\xrightarrow{\substack{R_1/4 \\ R_2 - R_1 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & 3/2 & -1/4 & 1/4 & -1/4 \\ 0 & 1/2 & 1/4 & 3/4 & -11/4 \\ 0 & -1 & 5/2 & -3/2 & 5/2 \end{bmatrix} \\ &\xrightarrow{\substack{2R_2 \\ R_3 + R_2 \\ R_3/3}} \begin{bmatrix} 1 & 3/2 & -1/4 & 1/4 & -1/4 \\ 0 & 1 & 1/2 & 3/2 & -11/2 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \end{aligned}$$

Therefore,

$$x_3 = x_5$$

$$x_2 = -\frac{1}{2}(x_3 + 3x_4 - 11x_5) = -\frac{1}{2}(3x_4 - 10x_5)$$

$$x_1 = -\frac{1}{4}(6x_2 - x_3 + x_4 - x_5) = -\frac{1}{4}(-9x_4 + 30x_5 - x_5 + x_4 - x_5) = 2x_4 - 7x_5$$

and

$$\mathbf{x} = \begin{bmatrix} 2x_4 - 7x_5 \\ -\frac{1}{2}(3x_4 - 10x_5) \\ x_5 \\ x_4 \\ x_5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ -3 \\ 0 \\ 2 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -7 \\ 5 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_5.$$

Therefore,

$$W^\perp = \text{span} \left(\begin{bmatrix} 4 \\ -3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

25 In \mathbb{R}^3 , let

- $W = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right),$
- $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$
- $\mathbf{w}' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and

$$\bullet \mathbf{v} = \mathbf{w} + \mathbf{w}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Clearly, \mathbf{w} and \mathbf{w}' are orthogonal but $\mathbf{w}' \notin W^\perp = \text{span} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

30 (a) Let $W = \text{span}(S)$. Then we can orthogonally decompose $\mathbf{x} \in \mathbb{R}^n$ as

$$\mathbf{x} = \mathbf{w} + \mathbf{w}^\perp$$

where $\mathbf{w} := \text{proj}_W(\mathbf{x})$ and $\mathbf{w}^\perp := \text{perp}_W(\mathbf{x})$. Also, since S is an orthonormal set, by definition on p.379,

$$\mathbf{w} = \sum_{i=1}^k (\mathbf{v}_i \cdot \mathbf{x}) \mathbf{v}_i.$$

Therefore,

$$\begin{aligned} \|\mathbf{x}\|^2 &= \mathbf{x} \cdot \mathbf{x} = (\mathbf{w} + \mathbf{w}^\perp) \cdot (\mathbf{w} + \mathbf{w}^\perp) \\ &= \mathbf{w}^\perp \cdot \mathbf{w}^\perp + \mathbf{w} \cdot \mathbf{w} \end{aligned}$$

since $\mathbf{w} \cdot \mathbf{w}^\perp = 0$.

Now,

$$\mathbf{w} \cdot \mathbf{w} = \left(\sum_{i=1}^k (\mathbf{v}_i \cdot \mathbf{w}) \mathbf{v}_i \right) \cdot \left(\sum_{i=1}^k (\mathbf{v}_i \cdot \mathbf{w}) \mathbf{v}_i \right) = \sum_{i=1}^k (\mathbf{v}_i \cdot \mathbf{x})^2 (\mathbf{v}_i \cdot \mathbf{v}_i) = \sum_{i=1}^k |\mathbf{v}_i \cdot \mathbf{x}|^2$$

since S is an orthonormal set. Therefore,

$$\|\mathbf{x}\|^2 = \|\mathbf{w}^\perp\|^2 + \sum_{i=1}^k |\mathbf{v}_i \cdot \mathbf{x}|^2 \geq \sum_{i=1}^k |\mathbf{v}_i \cdot \mathbf{x}|^2.$$

(b) In (a), the equality holds if and only if $\mathbf{w}^\perp = \mathbf{0}$, which is equivalent to satisfying $\mathbf{x} = \mathbf{w} = \text{proj}_W(\mathbf{x})$ and hence $\mathbf{x} \in \text{span}(S)$.

Exercise 5.3

$$10 \quad \text{(i)} \quad \mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

(ii)

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

(iii)

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore, the orthogonal vectors are

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

16 By normalizing the vectors, we get

$$Q = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & -1/\sqrt{6} & 1/\sqrt{2} \\ -1/2 & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/2 & 2/\sqrt{6} & 0 \end{bmatrix}$$

and since

$$R = Q^T A = \begin{bmatrix} 1/2 & 1/2 & -1/2 & 1/2 \\ 0 & -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2\sqrt{6} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix},$$

therefore

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} = QR = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & -1/\sqrt{6} & 1/\sqrt{2} \\ -1/2 & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/2 & 2/\sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 2\sqrt{6} & 0 \\ \sqrt{2} \end{bmatrix}$$

19 We don't need to apply the Gram-Schmidt process since A is already orthogonal. Since $R = Q^T A = A^T A = I$, the QR factorization results in $A = QR$ where $Q = A$ and $R = I$.

Excercise 5.4

10 (a) **Diagonalize** A ($A \rightarrow PDP^{-1}$)

The characteristic polynomial is, by the Laplace expansion theorem,

$$\begin{aligned}
 \det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & & & 1 \\ & 1 - \lambda & & \\ & & 1 - \lambda & \\ 1 & & & 2 - \lambda \end{vmatrix} \\
 &= (1 - \lambda)(-1)^{1+1} \begin{vmatrix} 2 - \lambda & & 1 \\ & 1 - \lambda & \\ & & 2 - \lambda \end{vmatrix} \\
 &= (1 - \lambda)^2(-1)^{1+1} \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} \\
 &= (1 - \lambda)^2((2 - \lambda)^2 - 1) \\
 &= (1 - \lambda)^2(\lambda^2 - 4\lambda + 3) = (\lambda - 1)^3(\lambda - 3)
 \end{aligned}$$

(i) For $\lambda_1 = 1$,

$$\text{The nullspace of } A - \lambda_1 I = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ is}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3$$

and hence

$$E_1 = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right).$$

(ii) For $\lambda_2 = 3$,

$$\text{The nullspace of } A - \lambda_2 I = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \text{ is}$$

$$\begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_1$$

and hence

$$E_3 = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Here, since all the eigenvectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are linearly independent, A can be diagonalizable, and

$$A = PDP^{-1}$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 3 \end{bmatrix}.$$

- (b) **Orthogonalize and normalize the eigenvectors by Gram-Schmidt process** ($P \rightarrow Q$)

Due to Theorem 5.19, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to all the vectors in E_1 .

Therefore we only need to orthogonalize the vectors in E_1 . But clearly, all the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

are orthogonal. So we only need to normalize the vectors to get Q as follows:

$$Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$$

- (c) **Apply Q to orthogonally diagonalize A** ($A \rightarrow QDQ^T$)

Summing up, $A = QDQ^T$ where

$$D = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 3 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$$

- 15 By the Spectral theorem, A and B are real symmetric matrices. Clearly, the entries of AB are also real numbers. Now, $(AB)^T = B^T A^T = BA = AB$ therefore AB is also a real symmetric matrices and hence orthogonally diagonalizable.

25 We need to show that $\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$ is orthogonal to \mathbf{q} . Now,

$$\mathbf{q} \cdot (\mathbf{v} - \text{proj}_W(\mathbf{v})) = \mathbf{q}^T (\mathbf{v} - \mathbf{q}\mathbf{q}^T\mathbf{v}) = \mathbf{q}^T\mathbf{v} - (\mathbf{q}^T\mathbf{q})\mathbf{q}^T\mathbf{v} = \mathbf{q}^T\mathbf{v} - \mathbf{q}^T\mathbf{v} = 0$$

where $\mathbf{q}^T\mathbf{q} = \mathbf{q} \cdot \mathbf{q} = 1$ since \mathbf{q} is a unit vector.

26 (a) We need to show that for $\mathbf{x} \in \mathbb{R}^n$, $\text{perp}_W(\mathbf{x}) = \mathbf{x} - \text{proj}_W(\mathbf{x})$ is orthogonal to all the vectors in $\{\mathbf{q}_1, \dots, \mathbf{q}_k\}$.

$$\begin{aligned} \mathbf{q}_i \cdot \text{perp}_W(\mathbf{x}) &= \mathbf{q}_i^T (\mathbf{x} - \text{proj}_W(\mathbf{x})) \\ &= \mathbf{q}_i^T \left(\mathbf{x} - \sum_{j=1}^k (\mathbf{q}_j \mathbf{q}_j^T) \mathbf{x} \right) \\ &= \mathbf{q}_i^T \mathbf{x} - \sum_{j=1}^k (\mathbf{q}_i^T \mathbf{q}_j \mathbf{q}_j^T) \mathbf{x} \\ &= \mathbf{q}_i^T \mathbf{x} - \sum_{j=1}^k (\mathbf{q}_i^T \mathbf{q}_j) \mathbf{q}_j^T \mathbf{x} \\ &= \mathbf{q}_i^T \mathbf{x} - (\mathbf{q}_i^T \mathbf{q}_i) \mathbf{q}_i^T \mathbf{x} \\ &= 0 \end{aligned}$$

since $\{\mathbf{q}_1, \dots, \mathbf{q}_k\}$ is an orthonormal set.

(b) Keeping in mind that $\{\mathbf{q}_1, \dots, \mathbf{q}_k\}$ is an orthonormal set,

$$\begin{aligned} P^T &= (\mathbf{q}_1 \mathbf{q}_1^T + \dots + \mathbf{q}_k \mathbf{q}_k^T)^T = (\mathbf{q}_1 \mathbf{q}_1^T)^T + \dots + (\mathbf{q}_k \mathbf{q}_k^T)^T \\ &= \mathbf{q}_1 \mathbf{q}_1^T + \dots + \mathbf{q}_k \mathbf{q}_k^T = P \end{aligned}$$

and

$$\begin{aligned} P^2 &= (\mathbf{q}_1 \mathbf{q}_1^T + \dots + \mathbf{q}_k \mathbf{q}_k^T)^2 = \mathbf{q}_1 \mathbf{q}_1^T \mathbf{q}_1 \mathbf{q}_1^T + \dots + \mathbf{q}_k \mathbf{q}_k^T \mathbf{q}_k \mathbf{q}_k^T \\ &= \mathbf{q}_1 \mathbf{q}_1^T + \dots + \mathbf{q}_k \mathbf{q}_k^T = P \end{aligned}$$

since

$$\mathbf{q}_i \mathbf{q}_i^T \mathbf{q}_j \mathbf{q}_j^T = \mathbf{q}_i (\mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_j^T = \begin{cases} \mathbf{q}_i \mathbf{q}_j^T & i = j \\ 0 & i \neq j \end{cases}$$

(c)

$$QQ^T = [\mathbf{q}_1 \quad \dots \quad \mathbf{q}_k] \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_k^T \end{bmatrix} = \mathbf{q}_1 \mathbf{q}_1^T + \dots + \mathbf{q}_k \mathbf{q}_k^T = P$$

By Theorem 3.25 and Theorem 3.28(a),

$$\text{rank}(P) = \text{rank}((QQ^T)) = \text{rank}(Q^T) = \text{rank}(Q) = k$$

since the columns of Q are orthogonal therefore linearly independent.