Homework #4 Solution

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Exercise 4.1

12 By the definition (p.253), λ is an eigenvalue of A if there is a nonzero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. This equation can be converted to $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$, therefore we can say that

"λ is an eigenvalue of A if the solution of the (homogeneous) linear system $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has a non-trivial solution. (Or if the nullity of $A - \lambda I$ is not zero.)

Assigning $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}.$$

To find the nullspace of $A - 2I$, we apply the Gaussian elimination as follows:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \xrightarrow[]{\substack{R_2 - R_1 \\ R_3 - 4R_1}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow[]{\substack{R_2/(-2) \\ R_3 + 2R_2}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathrm{rank}((A-2I)) = 2$, by the rank theorem, $\mathrm{nullity}((A-2I)) = 3-2 = 1 \neq 0$, therefore 2 is an eigenvalue of A.

37 The characteristic polynomial is

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - 0 \cdot b = (\lambda - a)(\lambda - d)$$

and hence the eigenvalues are a and d.

(a) For the eigenvalue a,

$$\det(A - aI) = \det\left(\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}\right).$$

The solution of the linear system $(A - aI)\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ therefore

$$E_a = \text{null}(A - aI) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue a.

(b) For the eigenvalue d,

$$\det{(A - dI)} = \det\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right).$$

The solution of the linear system $(A - dI)\boldsymbol{x} - \boldsymbol{0}$ is $\boldsymbol{x} = \begin{bmatrix} b \\ -a \end{bmatrix} t$ therefore

$$E_d = \text{null}(A - dI) = \text{span}\left(\begin{bmatrix} b \\ -a \end{bmatrix}\right)$$

and $\begin{bmatrix} b \\ -a \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue d.

Excercise 4.2

42 Let

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_i & \cdots & \boldsymbol{a}_j & \cdots & \boldsymbol{a}_n \end{bmatrix}$$

and

$$B = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_i & \cdots & k\boldsymbol{a}_i + \boldsymbol{a}_j & \cdots & \boldsymbol{a}_n \end{bmatrix},$$

where $\boldsymbol{a}_1 \cdots \boldsymbol{a}_n$ are column vectors. Now let C be the matrix obtained by replacing the j-th column of A with $k\boldsymbol{a}_i$. Then, B,C, and A are identical except that the jth column of B is the sum of the jth columns of C and A. Therefore by Theorem 4.3(e),

$$\det B = \det C + \det A.$$

Since $\det C = 0$ by Theorem 4.3(c), $\det B = \det A$.

The case with rows can be proved in the same way.

54

$$\det{(B^{-1}AB)} = \det B^{-1} \det A \det B = \frac{1}{\det B} \det A \det B = \det A.$$

65 Since A is invertible, by Theorem 4.12 and 4.7,

$$\det{(A^{-1})} = \frac{1}{\det A} = \det\left(\frac{1}{\det A} \operatorname{adj} A\right) = \frac{1}{(\det A)^n} \det{(\operatorname{adj} A)}$$

hence

$$\det(\operatorname{adj} A) = (\det A)^{n-1}.$$

Since A is invertible, $\det A \neq 0$ therefore $\det(\operatorname{adj} A) \neq 0$ and $\operatorname{adj} A$ is invertible, by Theorem 4.6.

By Theorem 4.12,

$$\begin{aligned}
(\operatorname{adj} A)^{-1} &= \left((\det A)A^{-1}\right)^{-1} \\
&= \frac{1}{\det A}(A^{-1})^{-1} && \text{(Theorem 3.9(b))} \\
&= \frac{1}{\det A}\left(\frac{1}{\det(A^{-1})}\operatorname{adj}(A^{-1})\right) && \text{(Theorem 4.12)} \\
&= \left(\frac{1}{\det A}\det A\right)\operatorname{adj}(A^{-1}) \\
&= \operatorname{adj}(A^{-1}).
\end{aligned}$$

70 (a) For the 4×4 matrix

$$A = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right],$$

A can be converted from I_4 by exchanging two pairs of columns (or rows). Therefore, $A = E_1E_2I_4$ and, by Theorem 4.4(a), $\det A = \det E_1 \det E_2 \det I_4 = 1$. But, since $\det P = \det S = 0$ and $\det Q = \det R = 1$, $(\det P)(\det S) - (\det Q)(\det R) = -1 \neq \det A$.

(b)

$$\begin{aligned}
BA &= \left[\begin{array}{c|c} P^{-1} & O \\ \hline -RP^{-1} & I \end{array}\right]\left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] \\
&= \left[\begin{array}{c|c} P^{-1}P + OR & P^{-1}Q + OS \\ \hline (-RP^{-1})P + IR & (-RP^{-1})Q + IS \end{array}\right] \\
&= \left[\begin{array}{c|c} I & P^{-1}Q \\ \hline O & -RP^{-1}Q + S \end{array}\right].
\end{aligned}$$

Therefore, by Excercise 69,

$$\det(BA) = (\det I)(\det(-RP^{-1}Q + S)) = \det(S - RP^{-1}Q).$$

On the other hand, by Theorem 4.10 and Excercise 69,

$$\det B = \det(B^T) = \det\left(\left[\begin{array}{c|c} (P^{-1})^T & (-RP^{-1})^T \\ \hline O & I \end{array}\right]\right) = (\det((P^{-1})^T))(\det I)$$
$$= \det(P^{-1}) = \frac{1}{\det P}.$$

Overall, since $\det(BA) = \det B \det A$,

$$\det A = \frac{\det(S - RP^{-1}Q)}{\det B} = \det P \det(S - RP^{-1}Q).$$

(c) From (b),

$$\begin{aligned}
\det A &= \det P \det(S - RP^{-1}Q) \\
&= \det(P(S - RP^{-1}Q)) && \text{(Theorem 4.8)} \\
&= \det(PS - PRP^{-1}Q) \\
&= \det(PS - RPP^{-1}Q) && (PR = RP) \\
&= \det(PS - RQ).
\end{aligned}$$

Excercise 4.3

12 (a) Note that

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 0 & 1 & 0 \\ 0 & 4-\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & 2 \\ 0 & 0 & 3 & -\lambda \end{bmatrix}.$$

By applying the Laplace expansion theorem with respect to the first column,

$$\det(A - \lambda I) = (4-\lambda)(-1)^{1+1}\det A_{11} = (4-\lambda)\begin{vmatrix} 4-\lambda & 1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 3 & -\lambda \end{vmatrix}.$$

Again, applying the theorem w.r.t. the first column,

$$\begin{aligned}
\begin{vmatrix} 4-\lambda & 1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 3 & -\lambda \end{vmatrix} &= (4-\lambda)(-1)^{1+1}\begin{vmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{vmatrix} = (4-\lambda)(-\lambda(1-\lambda) - 2\cdot 3) \\
&= (4-\lambda)(\lambda^2 - \lambda - 6) = (4-\lambda)(\lambda-3)(\lambda+2).
\end{aligned}$$

Therefore, the characteristic polynomial of A is

$$\det(A - \lambda I) = (\lambda - 4)^2(\lambda-3)(\lambda+2).$$

(b) The eigenvalues, which are the roots of the equation $\det(A - \lambda I) = 0$, are $\lambda_1 = 4$, $\lambda_2 = 3$, and $\lambda_3 = -2$.

(c) (i) For $\lambda_1 = 4$.
By applying the Gaussian elimination,

$$A - \lambda_1 I = A - 4I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 3 & -4 \end{bmatrix} \xrightarrow[R_4 - 3R_1]{\substack{R_2 - R_1 \\ R_3 + 3R_1}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

Since

$$\begin{aligned} x_3 = 0 &\to x_3 = 0 \\ x_4 = 0 &\to x_4 = 0, \end{aligned}$$

by taking free parameters t and s for x_1 and x_2 respectively, the eigenspace is

$$E_{\lambda_1} = E_4 = \left\{ \begin{bmatrix} t \\ s \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} s \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

(ii) For $\lambda_2 = 3$.

$$A - \lambda_2 I = A - 3I = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 3 & -3 \end{bmatrix} \xrightarrow[\ R_3 - 3R_3\ ]{R_3/(-2)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} 0 \cdot x_4 = 0 &\to x_4 = t \qquad\qquad \text{(free parameter)} \\ x_3 - x_4 = 0 &\to x_3 = t \\ x_2 + x_3 + x_4 = 0 &\to x_2 = -x_3 - x_4 = -2t \\ x_1 + x_3 = 0 &\to x_1 = -x_3 = -t \end{aligned}$$

and the eigenspace is

$$E_{\lambda_2} = E_3 = \left\{ \begin{bmatrix} -t \\ -2t \\ t \\ t \end{bmatrix} \right\} = \left\{ - \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix} t \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix} \right).$$

(iii) For $\lambda_3 = -2$.

$$A - \lambda_3 I = A + 2I = \begin{bmatrix} 6 & 0 & 1 & 0 \\ 0 & 6 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 3 & 2 \end{bmatrix} \xrightarrow{\substack{R_1/6 \\ R_2/6 \\ R_3/3 \\ R_4 - 3R_3}} \begin{bmatrix} 1 & 0 & 1/6 & 0 \\ 0 & 1 & 1/6 & 1/6 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} 0\cdot x_4 = 0 &\to x_4 = t \qquad\qquad \text{(free parameter)} \\ x_3 + \frac{2}{3}x_4 = 0 &\to x_3 = -\frac{2}{3}t \\ x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 = 0 &\to x_2 = -\frac{1}{9}t - \frac{1}{6}t = -\frac{5}{18}t \\ x_1 + \frac{1}{6}x_3 = 0 &\to x_1 = -\frac{1}{9}t \end{aligned}$$

and

$$E_{\lambda_3} = E_{-2} = \left\{ \begin{bmatrix} -(1/9)t \\ -(5/18)t \\ -(2/3)t \\ t \end{bmatrix} \right\} = \left\{ -\frac{1}{18} \begin{bmatrix} 2 \\ 5 \\ 12 \\ -18 \end{bmatrix} t \right\} = \text{span}\left(\begin{bmatrix} 2 \\ 5 \\ 12 \\ -18 \end{bmatrix} \right).$$

(d) Since $\dim E_4 = 2$, $\dim E_3 = 1$, and $\dim E_{-2} = 1$,

eigenvalue	4	3	-2
algebraic multiplicity	2	1	1
geometric multiplicity	2	1	1

17 From the condition, we have the equations

$$\begin{aligned} A\boldsymbol{v}_1 &= -\frac{1}{3}\boldsymbol{v}_1 \\ A\boldsymbol{v}_2 &= \frac{1}{3}\boldsymbol{v}_2 \\ A\boldsymbol{v}_3 &= \boldsymbol{v}_3 \end{aligned}$$

Since the solution of the linear system

$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

is

$$\begin{aligned} z = 2 &\to z = 2 \\ y + z = 1 &\to y = 1 - z = -1 \\ x + y + z = 2 &\to x = 2 - y - z = 1, \end{aligned}$$

we get

$$\boldsymbol{x} = \boldsymbol{v}_1 - \boldsymbol{v}_2 + 2\boldsymbol{v}_3.$$

Therefore,

$$A^{10}\boldsymbol{x} = A^{10}(\boldsymbol{v}_1 - \boldsymbol{v}_2 + 2\boldsymbol{v}_3) = A^{10}\boldsymbol{v}_1 - A^{10}\boldsymbol{v}_2 + 2A^{10}\boldsymbol{v}_3 = \left(-\frac{1}{3}\right)^{10}\boldsymbol{v}_1 - \left(\frac{1}{3}\right)^{10}\boldsymbol{v}_2 + 2\boldsymbol{v}_3$$

$$= \begin{bmatrix} \left(-\frac{1}{3}\right)^{10} - \left(\frac{1}{3}\right)^{10} + 2 \\ -\left(\frac{1}{3}\right)^{10} + 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{1}{3^{10}} + 2 \\ 2 \end{bmatrix}$$

18 It is straightforward to find

$$A^k\boldsymbol{x} = \begin{bmatrix} \frac{1}{(-3)^k} - \frac{1}{3^k} + 2 \\ -\frac{1}{3^k} + 2 \\ 2 \end{bmatrix}.$$

(a) k is even.

$$\lim_{k\to\infty} A^k\boldsymbol{x} = \lim_{k\to\infty} \begin{bmatrix} \frac{1}{3^k} - \frac{1}{3^k} + 2 \\ -\frac{1}{3^k} + 2 \\ 2 \end{bmatrix}. = \lim_{k\to\infty} \begin{bmatrix} 2 \\ -\frac{1}{3^k} + 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

(b) k is odd.

$$\lim_{k\to\infty} A^k\boldsymbol{x} = \lim_{k\to\infty} \begin{bmatrix} -\frac{1}{3^k} - \frac{1}{3^k} + 2 \\ -\frac{1}{3^k} + 2 \\ 2 \end{bmatrix}. = \lim_{k\to\infty} \begin{bmatrix} -\frac{2}{3^k} + 2 \\ -\frac{1}{3^k} + 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Therefore,

$$\lim_{k\to\infty} A^k\boldsymbol{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

22 $(A - cI)\boldsymbol{x} = A\boldsymbol{x} - c\boldsymbol{x} = \lambda\boldsymbol{x} - c\boldsymbol{x} = (\lambda - c)\boldsymbol{x}$.

41 (a) Sum of eigenvalues.

From 40, $\operatorname{tr}(A)$ and $\operatorname{tr}(B)$ are each the sum of the eigenvalues of A and B, respectively. On the other hand, from the excercise 44(a) of Chap. 3.2 (p.160),

$$\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B).$$

(b) Product of eigenvalues.

This can be easily proved since $\det AB = \det A \det B$ and from 40, $\det A$ and $\det B$ are each the product of all the eigenvalue of A and B, respectively.

Excercise 4.4

11 The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\ &= (1-\lambda)(-1)^{1+1}\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + (-1)^{1+3}\begin{vmatrix} 0 & 1-\lambda \\ 1 & 1 \end{vmatrix} \\ &= (1-\lambda)(-\lambda(1-\lambda)-1)-(1-\lambda) \\ &= (1-\lambda)(\lambda^2-\lambda-1)-1+\lambda \\ &= -\lambda^3+2\lambda^2+\lambda-2 \\ &= (1-\lambda)(\lambda^2-\lambda-2) \\ &= (1-\lambda)(\lambda-2)(\lambda+1)\end{aligned}$$

(a) For $\lambda_1 = 1$.

$$A-\lambda_1 I = A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[]{\substack{R_1 \leftrightarrow R_3 \\ R_3 - R_2}} \begin{bmatrix} 1 & 1 & -10 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the solution of the homogeneous linear system $A\boldsymbol{x} = \mathbf{0}$ is

$$\begin{aligned} z = 0 &\to z = 0 \\ x + y - z = 0 &\to x = t, y = -x = -t \end{aligned}$$

and hence

$$E_{\lambda_1} = E_1 = \left\{\begin{bmatrix} t \\ -t \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} t\right\} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right).$$

(b) For $\lambda_2 = 2$.

$$
\begin{aligned}
A - \lambda_2 I &= A - 2I \\
&= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\
&\xrightarrow[\quad]{\substack{-R_1 \\ R_3 - R_1}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\
&\xrightarrow[\quad]{\substack{-R_2 \\ R_3 - R_2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 \cdot z = 0 &\to z = t && \text{(free parameter)} \\
y - z = 0 &\to y = t \\
x - z = 0 &\to x = t
\end{aligned}
$$

and

$$
E_{\lambda_2} = E_2 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right).
$$

(c) For $\lambda_3 = -1$.

$$
\begin{aligned}
A - \lambda_3 I &= A + I \\
&= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\xrightarrow[\quad]{\substack{R_1/2 \\ R_3 - R_1}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 0 & 1 & 1/2 \end{bmatrix}
$$

$$
\xrightarrow[\quad]{\substack{R_2/2 \\ R_3 - R_2}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}
$$

therefore

$$
\begin{aligned}
0 \cdot z = 0 &\to z = t && \text{(free parameter)} \\
y + \frac{1}{2}z = 0 &\to y = -\frac{1}{2}t \\
x + \frac{1}{2}z = 0 &\to x = -\frac{1}{2}t
\end{aligned}
$$

and

$$E_{\lambda_3} = E_{-1} = \left\{ \begin{bmatrix} -\frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix} \right\} = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right).$$

Therefore,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 2 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}^{-1}.$$

13 First we need to find the eigenvalues. The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\ &= (1-\lambda)(-1)^{1+1} \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 1 & -\lambda \end{vmatrix} + (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ -\lambda & 1 \end{vmatrix} \\ &= (1-\lambda)(\lambda^2 - 1) + (-2\lambda - 1) + (2 + \lambda) \\ &= -\lambda^3 + \lambda^2 = \lambda^2(1-\lambda). \end{aligned}$$

(a) For $\lambda_1 = 0$.

$$\begin{aligned} A - \lambda_1 I = A &= \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &\xrightarrow[R_3 - R_1]{R_2 + R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix} \\ &\xrightarrow[R_3 + R_2]{R_2/2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 \cdot z = 0 &\rightarrow z = t \qquad \text{(free parameter)} \\ y + z = 0 &\rightarrow y = -t \\ z + 2y + z = 0 &\rightarrow x = -2y - z = t \end{aligned}$$

and hence

$$E_{\lambda_1} = E_0 = \left\{ \begin{bmatrix} t \\ -t \\ t \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} t = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

Since the geometric multiplicity of the eigenvalue $\lambda_1 = 0$ is 1, while its algebraic multiplicity is 2, A is not diagonalizable.

32 We can prove by showing that $\text{nullity}(A) = \text{nullity}(B)$ due to the rank theorem (p.203). Let $AP = PB$.

(a) We first prove that $\text{nullity}(A) \leq \text{nullity}(B)$ by showing that for any $\boldsymbol{x} \in \text{null}(A)$, there exists a unique vector $P^{-1}\boldsymbol{x}$ in $\text{null}(B)$.
Let $\boldsymbol{x} \in \text{null}(A)$ and $\boldsymbol{x} \neq \mathbf{0}$. Then,

$$A\boldsymbol{x} = (PBP^{-1})\boldsymbol{x} = P(B(P^{-1}\boldsymbol{x})) = \mathbf{0}.$$

Since P is invertible, $P^{-1}\boldsymbol{x} \neq \mathbf{0}$ and $P^{-1}\boldsymbol{x}$ is unique for given $\boldsymbol{x}$. Also, since P is invertibble, $P(B(P^{-1}\boldsymbol{x})) = \mathbf{0}$ if and only if $B(P^{-1}\boldsymbol{x}) = \mathbf{0}$. Therefore $P^{-1}\boldsymbol{x} \in \text{null}(B)$.

(b) We can also prove that $\text{nullity}(B) \leq \text{nullity}(A)$ in the same way by showing that for any $\boldsymbol{x} \in \text{null}(B)$, there exists a unique vector $P\boldsymbol{x}$ in $\text{null}(A)$.

Overall, $\text{nullity}(A) = \text{nullity}(B)$.

34 With $P = A$,

$$P^{-1}(AB)P = A^{-1}ABA = BA.$$

40 Let

$$P^{-1}AP = B.$$

With $Q = (P^T)^{-1}$,

$$Q^{-1}A^TQ = (Q^TA(Q^{-1})^T)^T = P^{-1}AP^T = B^T.$$

43 If we diagonalize A as

$$A = PDP^{-1},$$

since $D = \lambda I$,

$$A = P(\lambda I)P^{-1} = \lambda(PIP^{-1}) = \lambda I.$$

45 This is true due to 4.22(e).