

# Homework #3 Solution

Minho Kim

April 12, 2010

## Excercise 3.1

19 Below is the data given in the problem.

	W.H. #1	W.H. #2		doohickies	gizmos	widgets
doohichies	200	75	by truck	\$1.50	\$1.00	\$2.00
gizmos	150	100	by train	\$1.75	\$1.50	\$1.00
widgets	100	125				

(a) (b)

Table 1: (a) Number of units of each product shipped to each warehouse and (b) cost of shipping one unit of each product by each transportation.

Let

$$A = \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$$

then the cost of shipping products to each warehouse can be computed as

$$BA = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix} \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix}$$

	warehouse 1	warehouse 2
by truck	\$650.00	\$462.50
by train	\$675.00	\$406.25

Table 2: Cost of shipping products to each warehouse.

29 The statement “columns of  $B$  are linearly dependent” is equivalent to the statement “the linear system  $B\mathbf{x} = \mathbf{0}$  has a non-trivial solution.” Now

let  $\mathbf{u} \neq \mathbf{0}$  be a non-trivial solution of  $B\mathbf{x} = \mathbf{0}$ . Then, since  $(AB)\mathbf{u} = A(B\mathbf{u}) = \mathbf{0}$ ,  $\mathbf{u}$  is a non-trivial solution of the linear system  $(AB)\mathbf{x} = \mathbf{0}$ . Therefore the columns of  $AB$  are linearly dependent.

38 Note that, due to the trigonometric equations, (See [http://en.wikipedia.org/wiki/Trigonometric\\_equation](http://en.wikipedia.org/wiki/Trigonometric_equation))

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta.\end{aligned}$$

(a)  $A^2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^2 = \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -2 \cos \theta \sin \theta \\ 2 \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}$ . By the trigonometric equations,

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta\end{aligned}$$

therefore

$$A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

(b) First, it is straightforward to show that the equation holds for  $n = 1$ . Assuming the equation holds for  $n = k$ ,

$$\begin{aligned}A^{k+1} &= A^k A = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos k\theta \cos \theta - \sin k\theta \sin \theta & -\cos k\theta \sin \theta - \sin k\theta \cos \theta \\ \sin k\theta \cos \theta + \cos k\theta \sin \theta & -\sin k\theta \sin \theta + \cos k\theta \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}\end{aligned}$$

due to the trigonometric equation above and the proof is complete.

### Exercise 3.2

27 Let  $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ .

$$(a) AB = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{bmatrix}.$$

$$(b) BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}.$$

For  $B$  to commute to any matrix  $A$ , the equation  $AB = BA$  must hold for any  $x, y, z, w \in \mathbb{R}$ . In other words,

$$\begin{aligned}\begin{bmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{bmatrix} &= \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix} \\ \rightarrow \begin{bmatrix} cy - bz & bx + (d - a)y - bw \\ -cx + cw + (a - d)z & -cy + bz \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

for any  $x, y, z, w \in \mathbb{R}$ . The solution is therefore  $c = 0$ ,  $d = 0$  and  $a = d$ . In other words, the matrix  $B$  must be a scalar multiple of identity matrix.

- 29 Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are both upper triangular matrices and  $C = [c_{ij}] = AB$ . Due to the definition of matrix multiplication,

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}.$$

For  $i > j$ ,

- (a)  $a_{ik} = 0$  for  $1 \leq k < i$  due to the definition of triangular matrices, therefore

$$\sum_{k=1}^{i-1} a_{ik}b_{kj} = 0.$$

- (b)  $b_{kj} = 0$  for  $i \leq k \leq n$  due to the definition of triangular matrices, therefore

$$\sum_{k=i}^n a_{ik}b_{kj} = 0.$$

Therefore, for  $i > j$ ,

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^n a_{ik}b_{kj} = 0$$

hence  $C = [c_{ij}]$  is an upper triangular matrix.

- 33 First, the equality holds for  $n = 1$  since

$$(A_1)^T = A_1^T.$$

Assume that the equality holds for  $n = k$ , i.e.,

$$(A_1 \cdots A_k)^T = A_k^T \cdots A_1^T.$$

Then

$$\begin{aligned} (A_1 \cdots A_k A_{k+1})^T &= ((A_1 \cdots A_k)A_{k+1})^T = A_{k+1}^T (A_1 \cdots A_k)^T = A_{k+1}^T (A_k^T \cdots A_1^T) \\ &= A_{k+1}^T A_k^T \cdots A_1^T \end{aligned}$$

therefore the equality holds for all  $n$ .

- 42

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$$

47 By the Exercise 44(a) and 45,

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0$$

but

$$\operatorname{tr}(I_2) = 2$$

therefore the equality cannot hold.

Exercise 3.3

23

$$\begin{aligned} ABXA^{-1}B^{-1} &= I + A \\ \rightarrow ABXA^{-1} &= (I + A)B \\ \rightarrow ABX &= (I + A)BA \\ \rightarrow BX &= A^{-1}(I + A)BA \\ \rightarrow X &= B^{-1}A^{-1}(I + A)BA = (B^{-1}A^{-1} + B^{-1})BA = B^{-1}A^{-1}BA + A \end{aligned}$$

44(b)

$$A^2 = A \rightarrow A^2A^{-1} = AA^{-1} \rightarrow A = I.$$

46

$$\begin{aligned} (A^{-1})^T &= (A^T)^{-1} && \text{(Theorem 3.9(d))} \\ &= A^{-1} && \text{(Since } A = A^T) \end{aligned}$$

therefore  $A^{-1}$  is symmetric.

47 Note that Theorem 3.9(c) requires that both  $A$  and  $B$  are invertible. Therefore we cannot directly use Theorem 3.9(a).

(a) Since  $AB$  is invertible,

$$(AB)(AB)^{-1} = I \rightarrow A(B(AB)^{-1}) = I$$

therefore there is a matrix,  $C := B(AB)^{-1}$ , such that

$$AC = I.$$

By Theorem 3.13,  $A$  is invertible.

(b) Again, since  $AB$  is invertible,

$$(AB)^{-1}(AB) = I \rightarrow ((AB)^{-1}A)B = I$$

therefore there is a matrix,  $D := (AB)^{-1}A$ , such that

$$DB = I.$$

By Theorem 3.13,  $B$  is invertible.

67

$$\begin{bmatrix} O & B \\ C & D \end{bmatrix} \begin{bmatrix} -(BD^{-1}C)^{-1} & (BD^{-1}C)^{-1}BD^{-1} \\ D^{-1}C(BD^{-1}C)^{-1} & D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} = \\ \begin{bmatrix} BD^{-1}C(BD^{-1}C)^{-1} & B(D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) \\ -C(BD^{-1}C)^{-1} + DD^{-1}C(BD^{-1}C)^{-1} & C(BD^{-1}C)^{-1}BD^{-1} + D(D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) \end{bmatrix}$$

(a)

$$BD^{-1}C(BD^{-1}C)^{-1} = (BD^{-1}C)(BD^{-1}C)^{-1} = I.$$

(b)

$$\begin{aligned} B(D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) &= BD^{-1} - BD^{-1}C(BD^{-1}C)^{-1}BD^{-1} \\ &= BD^{-1} - (BD^{-1}C)(BD^{-1}C)^{-1}BD^{-1} \\ &= BD^{-1} - BD^{-1} = O. \end{aligned}$$

(c)

$$-C(BD^{-1}C)^{-1} + DD^{-1}C(BD^{-1}C)^{-1} = -C(BD^{-1}C)^{-1} + C(BD^{-1}C)^{-1} = O.$$

(d)

$$\begin{aligned} &C(BD^{-1}C)^{-1}BD^{-1} + D(D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) \\ &= C(BD^{-1}C)^{-1}BD^{-1} + I - DD^{-1}C(BD^{-1}C)^{-1}BD^{-1} \\ &= I. \end{aligned}$$

Therefore the result is

$$\begin{bmatrix} I & O \\ O & I \end{bmatrix} = I.$$

Excercise 3.5

20

$$\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{\begin{array}{l} R_1/2 \\ R_2 + R_1 \\ R_3 - R_1 \end{array}} \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 1 & 3 & 7/2 \end{bmatrix} \\ \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) From the reduced row echelon form, a basis for row( $A$ ) is

$$\{[1 \ -2 \ 0 \ 1 \ 1/2], [0 \ 0 \ 1 \ 3 \ 7/2]\}.$$

- (b) Since the 1st and 3rd column has leading 1's, a basis for  $\text{col}(A)$  is the set of the 1st and 3rd column vectors of  $A$ :

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- (c) Since the solution of the linear system  $A\mathbf{x} = \mathbf{0}$  is

$$\begin{aligned} x_1 &= 2x_2 - x_4 - \frac{1}{2}x_5 \\ x_3 &= -3x_4 - \frac{7}{2}x_5, \end{aligned}$$

with the free parameters  $x_2 = s$ ,  $x_4 = t$ , and  $x_5 = u$ ,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 4t - \frac{1}{2}u \\ s \\ -3t - \frac{7}{2}u \\ t \\ u \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix} u.$$

Therefore, a basis for  $\text{null}(A)$  is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -7 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

43

$$\begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \xrightarrow{\begin{array}{l} R_2 + 2R_1 \\ R_3 - aR_1 \end{array}} \begin{bmatrix} 1 & 2 & a \\ 0 & 4a + 4 & 2 + 2a \\ 0 & -2 - 2a & 1 - a^2 \end{bmatrix}$$

- (a) If  $a \neq -1$ ,

$$\xrightarrow{\begin{array}{l} R_2/4(a+1) \\ R_1 - 2R_2 \\ R_3 + 2(a+1)R_2 \end{array}} \begin{bmatrix} 1 & 0 & a-1 \\ 0 & 1 & 1/2 \\ 0 & 0 & (2-a)(1+a) \end{bmatrix}$$

therefore

$$\text{rank}(A) = \begin{cases} 2 & \text{if } a = 2 \\ 3 & \text{if } a \neq 2 \end{cases}$$

(b) If  $a = -1$ ,

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

therefore  $\text{rank}(A) = 1$

Overall,

$$\text{rank}(A) = \begin{cases} 1 & \text{if } a = -1 \\ 2 & \text{if } a = 2 \\ 3 & \text{if } a \notin \{-1, 2\}. \end{cases}$$

50 We need to find the solution of the linear system

$$\begin{bmatrix} 3 & 5 \\ 1 & 1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Since

$$\begin{aligned} \begin{bmatrix} 3 & 5 & | & 1 \\ 1 & 1 & | & 3 \\ 4 & 6 & | & 4 \end{bmatrix} &\xrightarrow{\substack{R_1/3 \\ R_2 - R_1 \\ R_3 - 4R_1}} \begin{bmatrix} 1 & 5/3 & | & 1/3 \\ 0 & -2/3 & | & 8/3 \\ 0 & -2/3 & | & 8/3 \end{bmatrix} \\ &\xrightarrow{\substack{R_2/(-2/3) \\ R_1 - 5/3R_2 \\ R_3 + (2/3)R_2}} \begin{bmatrix} 1 & 0 & | & 7 \\ 0 & 1 & | & -4 \\ 0 & 0 & | & 0 \end{bmatrix} \end{aligned}$$

therefore,

$$7 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - 4 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \mathbf{w}$$

and

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

55 Due to the definition of null space, for any column vector  $\mathbf{v} \in \text{null}(A)$ ,

$$A\mathbf{v} = \mathbf{0}.$$

A row vector  $\mathbf{u} \in \text{row}(A)$  can be written as

$$\mathbf{u} = \mathbf{y}A$$

for some  $\mathbf{y} \in \mathbb{R}^m$ . Since

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v} = (\mathbf{y}A)\mathbf{v} = \mathbf{y}(A\mathbf{v}) = \mathbf{y}\mathbf{0} = 0,$$

the proof is complete.

57(a) Let  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times k}$  hence  $AB \in \mathbb{R}^{m \times k}$ .

For any vector  $\mathbf{u} \in \text{null}(B)$ ,  $B\mathbf{u} = \mathbf{0}$ , hence  $(AB)\mathbf{u} = A(B\mathbf{u}) = A\mathbf{0} = \mathbf{0}$ . Therefore,  $\text{null}(B) \subset \text{null}(AB)$  hence  $\text{nullity}(B) \leq \text{nullity}(AB)$ . Due to The Rank Theorem,

$$\text{rank}(B) + \text{nullity}(B) = k = \text{rank}(AB) + \text{nullity}(AB).$$

Since  $\text{nullity}(B) \leq \text{nullity}(AB)$ ,  $\text{rank}(B) \geq \text{rank}(AB)$  and the proof is complete.

59 (a) (i) Due to the Exercise 57(a),

$$\text{rank}(UA) \leq \text{rank}(A).$$

(ii) Due to the Exercise 57(a),

$$\text{rank}(A) = \text{rank}(U^{-1}(UA)) \leq \text{rank}(UA).$$

Therefore  $\text{rank}(A) \leq \text{rank}(UA) \leq \text{rank}(A)$  hence  $\text{rank}(UA) = \text{rank}(A)$ .

(b) (i) Due to the Exercise 58(a),

$$\text{rank}(AV) \leq \text{rank}(A).$$

(ii) Due to the Exercise 58(a),

$$\text{rank}(A) = \text{rank}((AV)V^{-1}) \leq \text{rank}(AV).$$

Therefore,  $\text{rank}(A) = \text{rank}(AV)$ .

60 Note that we need the condition that  $A \neq O$  (zero matrix)

(a)  $A = \mathbf{u}\mathbf{v}^T$  with  $\mathbf{u} \in \mathbb{R}^m$  and  $\mathbf{v} \in \mathbb{R}^n \Rightarrow \text{rank}(A) = 1$

Due to Exercise 58(a),

$$\text{rank}(A) = \text{rank}(\mathbf{u}\mathbf{v}^T) \leq \text{rank}(\mathbf{u}) = 1.$$

Since  $A \neq O$ ,  $\text{rank}(A) \geq 1$ . Therefore,  $\text{rank}(A) = 1$ .

(b)  $\text{rank}(A) = 1 \Rightarrow A = \mathbf{u}\mathbf{v}^T$  with  $\mathbf{u} \in \mathbb{R}^m$  and  $\mathbf{v} \in \mathbb{R}^n$

Since  $\text{rank}(A) = 1$ , there is only one vector in the basis of  $\text{row}(A)$  hence all the rows of  $A$  are scalar multiples of a row vector, say,  $\mathbf{x} \in \mathbb{R}^n$ . Therefore,

$$A = \begin{bmatrix} c_1\mathbf{x} \\ \vdots \\ c_m\mathbf{x} \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \mathbf{x}$$

hence

$$A = \mathbf{u}\mathbf{v}^T$$

where

$$\mathbf{u} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \mathbf{x}^T.$$

Excercise 3.6

17 By

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

a vector is transformed as follows:

$$D\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}.$$

18 Due to Example 3.59 (p.216-217), the projection matrix ( $d_1 = d_2 = 1$ )

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

26 (b) From the parallelogram in Figure 3.14, we can see that

$$\mathbf{x} + F_l(\mathbf{x}) = 2P_l(\mathbf{x})$$

therefore

$$F_l(\mathbf{x}) = 2P_l(\mathbf{x}) - \mathbf{x}.$$

Due to Example 3.59,

$$[P_l] = \frac{1}{d_1^2 + d_2^2} \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

therefore

$$\begin{aligned} [F_l] &= 2[P_l] - I = \frac{2}{d_1^2 + d_2^2} \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{d_1^2 + d_2^2} \begin{bmatrix} 2d_1^2 - (d_1^2 + d_2^2) & 2d_1 d_2 \\ 2d_1 d_2 & 2d_2^2 - (d_1^2 + d_2^2) \end{bmatrix} \\ &= \frac{1}{d_1^2 + d_2^2} \begin{bmatrix} d_1^2 - d_2^2 & 2d_1 d_2 \\ 2d_1 d_2 & -d_1^2 + d_2^2 \end{bmatrix}. \end{aligned}$$

(c) We can decompose the transformation as the following three steps:

(i) Rotate both the line  $l$  and the vector  $\mathbf{x}$  by the angle  $-\theta$  such that the line  $l$  coincides with the  $x$ -axis. Let

$$R_1 := \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

be the rotation matrix.

- (ii) Reflect rotated vector  $R_{-\theta}\mathbf{x}$  with respect to the  $x$ -axis. Due to the Example 3.56, the reflection matrix is

$$F := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- (iii) Rotate the reflected vector by the angle  $\theta$ . Let

$$R_2 := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

be the rotation matrix.

Therefore

$$\begin{aligned} [F_l] &= R_2 F R_1 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & 2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \sin^2 \theta - \cos^2 \theta \end{bmatrix}. \end{aligned}$$

Due to the trigonometric equations,

$$[F_l] = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

40 Again, the trigonometric equations says

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta. \end{aligned}$$

Therefore,

$$\begin{aligned} [R_\alpha][R_\beta] &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -(\cos \alpha \sin \beta + \sin \alpha \cos \beta) \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} \\ &= [R_{\alpha+\beta}]. \end{aligned}$$

41 Let  $\theta_m$  be the angle between  $m$  and the positive  $x$ -axis, and let  $\theta_l$  be the angle between  $l$  and the positive  $x$ -axis. Then, due to the Exercise 26(c),

$$[F_m] = \begin{bmatrix} \cos 2\theta_m & \sin 2\theta_m \\ \sin 2\theta_m & -\cos 2\theta_m \end{bmatrix} \quad \text{and} \quad [F_l] = \begin{bmatrix} \cos 2\theta_l & \sin 2\theta_l \\ \sin 2\theta_l & -\cos 2\theta_l \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
[F_m \circ F_l] &= [F_m][F_l] = \begin{bmatrix} \cos 2\theta_m & \sin 2\theta_m \\ \sin 2\theta_m & -\cos 2\theta_m \end{bmatrix} \begin{bmatrix} \cos 2\theta_l & \sin 2\theta_l \\ \sin 2\theta_l & -\cos 2\theta_l \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\theta_m \cos 2\theta_l + \sin 2\theta_m \sin 2\theta_l & \cos 2\theta_m \sin 2\theta_l - \sin 2\theta_m \cos 2\theta_l \\ \sin 2\theta_m \cos 2\theta_l - \cos 2\theta_m \sin 2\theta_l & \sin 2\theta_m \sin 2\theta_l + \cos 2\theta_m \cos 2\theta_l \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\theta_m - \theta_l) & \sin 2(-\theta_m + \theta_l) \\ \sin 2(\theta_m - \theta_l) & \cos 2(\theta_m - \theta_l) \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\theta_m - \theta_l) & -\sin 2(\theta_m - \theta_l) \\ \sin 2(\theta_m - \theta_l) & \cos 2(\theta_m - \theta_l) \end{bmatrix} = [R_{2(\theta_m - \theta_l)}].
\end{aligned}$$

Since  $\theta = \theta_m - \theta_l$ , the equality holds.

54 Let  $R_T$  be the range of the linear transformation  $T$ . We show that  $R_T \subset \text{col}([T])$  and  $\text{col}([T]) \subset R_T$ . Let  $\mathbf{t}_1, \dots, \mathbf{t}_n$  be the columns of  $[T]$  such that

$$[T] = [ \mathbf{t}_1 \mid \cdots \mid \mathbf{t}_n ].$$

(a)  $R_T \subset \text{col}([T])$

For any vector  $\mathbf{x} \in \mathbb{R}^n$ ,

$$[T]\mathbf{x} = [ \mathbf{t}_1 \mid \cdots \mid \mathbf{t}_n ] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{t}_1 + \cdots + x_n\mathbf{t}_n \in \text{col}([T])$$

(b)  $\text{col}([T]) \subset R_T$

Due to the definition of the column space, any vector  $\mathbf{u} \in \text{col}([T])$  can be written as a linear combination of the columns of  $[T]$ :

$$\mathbf{u} = y_1\mathbf{t}_1 + \cdots + y_n\mathbf{t}_n$$

therefore

$$\mathbf{u} = [ \mathbf{t}_1 \mid \cdots \mid \mathbf{t}_n ] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in R_T.$$