

# Homework #1 Solution

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Exercise 1.2

58 (a) We need to prove both “if” and “only if” statements.

(i) “if” statement:  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal  $\Rightarrow \|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$

Since  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal,  $\mathbf{u} \cdot \mathbf{v} = 0$ . Firstly,

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})} = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}} = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}.$$

Secondly,

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}} = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}.$$

Therefore,  $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$ .

(ii) “only if” statement:  $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\| \Rightarrow \mathbf{u}$  and  $\mathbf{v}$  are orthogonal

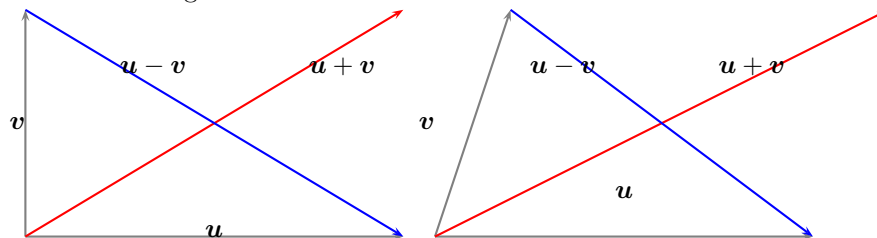
By squaring both sides, we get

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &= \|\mathbf{u} - \mathbf{v}\|^2 \\ (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} \\ 4\mathbf{u} \cdot \mathbf{v} &= 0\end{aligned}$$

therefore  $\mathbf{u} \cdot \mathbf{v} = 0$  hence  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal.

(b) From the pictures below, we can conclude that

“Two diagonals of a parallelogram have the same lengths **if and only if** it is a rectangle.”



65 (a)

$$\begin{aligned} & (u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1v_1 + u_2v_2)^2 \\ &= [(u_1v_1)^2 + (u_1v_2)^2 + (u_2v_1)^2 + (u_2v_2)^2] - [(u_1v_1)^2 + (u_2v_2)^2 + 2u_1v_1u_2v_2] \\ &= (u_1v_2)^2 + (u_2v_1)^2 - 2u_1v_1u_2v_2 \\ &= (u_1v_2 - u_2v_1)^2 \geq 0 \end{aligned}$$

therefore

$$(u_1v_1 + u_2v_2)^2 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2).$$

(b) The analogue in  $\mathbb{R}^3$  is

$$(u_1v_1 + u_2v_2 + u_3v_3)^2 \leq (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2).$$

We can prove it using the same method:

$$\begin{aligned} & (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &= [(u_1v_1)^2 + (u_1v_2)^2 + (u_1v_3)^2 + (u_2v_1)^2 + (u_2v_2)^2 + (u_2v_3)^2 + (u_3v_1)^2 + (u_3v_2)^2 + (u_3v_3)^2] \\ &\quad - [(u_1v_1)^2 + (u_2v_2)^2 + (u_3v_3)^2 + 2(u_1v_1u_2v_2 + u_2v_2u_3v_3 + u_3v_3u_1v_1)] \\ &= [(u_1v_2)^2 + (u_2v_1)^2 - 2(u_1v_1u_2v_2)] + [(u_2v_3)^2 + (u_3v_2)^2 - 2(u_2v_2u_3v_3)] \\ &\quad + [(u_3v_1)^2 + (u_1v_3)^2 - 2(u_3v_3u_1v_1)] \\ &= (u_1v_2 - u_2v_1)^2 + (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 \geq 0. \end{aligned}$$

66

$$\begin{aligned} \|\text{proj}_{\mathbf{u}}(\mathbf{v})\| &= \left\| \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} \right\| \\ &= \left| \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \right| \|\mathbf{u}\| \\ &= \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|^2} \|\mathbf{u}\| \\ &= \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|} \\ &\leq \|\mathbf{v}\| \end{aligned}$$

therefore

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

67 Since  $\mathbf{u}$  and  $\mathbf{v} - c\mathbf{u}$  are orthogonal,

$$\mathbf{u} \cdot (\mathbf{v} - c\mathbf{u}) = 0.$$

Therefore,

$$\mathbf{u} \cdot \mathbf{v} - c\mathbf{u} \cdot \mathbf{u} = 0$$

hence

$$c = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}.$$

68 First, note that that, for any vectors  $\mathbf{u}$  and  $\mathbf{v}$ ,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Proof:

$$\begin{aligned} (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 - \|\mathbf{u} + \mathbf{v}\|^2 &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 - (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| - [\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}] \\ &= 2(\|\mathbf{u}\|\|\mathbf{v}\| - \mathbf{u} \cdot \mathbf{v}) \geq 0 \end{aligned}$$

since, due to the Cauchy-Schwarz Inequality,

$$\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

For  $n = 1$ , the inequality becomes

$$\|\mathbf{v}_1\| \leq \|\mathbf{v}_1\|$$

which is true.

Assuming the inequality is satisfied for  $n = k$ , i.e.,

$$\|\mathbf{v}_1 + \cdots + \mathbf{v}_k\| \leq \|\mathbf{v}_1\| + \cdots + \|\mathbf{v}_k\|.$$

According to the inequality proved above,

$$\|\mathbf{v}_1 + \cdots + \mathbf{v}_k + \mathbf{v}_{k+1}\| \leq \|\mathbf{v}_1 + \cdots + \mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|.$$

Since

$$\|\mathbf{v}_1 + \cdots + \mathbf{v}_k\| \leq \|\mathbf{v}_1\| + \cdots + \|\mathbf{v}_k\|$$

by the assumption, we get

$$\|\mathbf{v}_1 + \cdots + \mathbf{v}_k + \mathbf{v}_{k+1}\| \leq \|\mathbf{v}_1 + \cdots + \mathbf{v}_k\| + \|\mathbf{v}_{k+1}\| \leq \|\mathbf{v}_1\| + \cdots + \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|$$

therefore the inequality is satisfied for  $n = k + 1$  the proof is complete.

### Exercise 1.3

26 Let

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is the point which is equidistant from two points

$$P = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} x_q \\ y_q \\ z_q \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

Then we have the equation

$$(x - x_p)^2 + (y - y_p)^2 + (z - z_p)^2 = (x - x_q)^2 + (y - y_q)^2 + (z - z_q)^2$$

which can be converted to

$$(x_p - x_q)x + (y_p - y_q)y + (z_p - z_q)z = \frac{1}{2} [x_p^2 + y_p^2 + z_p^2 - (x_q^2 + y_q^2 + z_q^2)]$$

$$\mathbf{x} \cdot (\mathbf{p} - \mathbf{q}) = \frac{1}{2} (\|\mathbf{p}\|^2 - \|\mathbf{q}\|^2)$$

where  $\mathbf{x}$ ,  $\mathbf{p}$  and  $\mathbf{q}$  are the point vectors corresponding to  $X$ ,  $P$  and  $Q$ , respectively. Substituting the values for  $P$  and  $Q$ , we get

$$\mathbf{x} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 10.$$

42 (This proof is different from the one I gave in the class.)

First, note that distance equation on p.41 can be converted to

$$\frac{|\mathbf{n} \cdot \mathbf{p} - d|}{\|\mathbf{p}\|}$$

where

$$\mathbf{n} := \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad \mathbf{p} := \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}.$$

Let  $\mathbf{p}$  be the point vector on the plane  $\mathbf{n} \cdot \mathbf{x} = d_1$ . Then the distance from  $\mathbf{p}$  to the plane  $\mathbf{n} \cdot \mathbf{x} = d_2$  is given as

$$\frac{|\mathbf{n} \cdot \mathbf{p} - d_2|}{\|\mathbf{n}\|}.$$

Since  $\mathbf{p}$  is on the plane  $\mathbf{n} \cdot \mathbf{x} = d_1$ , we have  $\mathbf{n} \cdot \mathbf{p} = d_1$  therefore the distance is

$$\frac{|d_1 - d_2|}{\|\mathbf{n}\|}.$$

47 Since  $\mathbf{n}$  is orthogonal to every vector  $\mathbf{p} = \mathbf{v} - c\mathbf{n} \in \mathcal{P}$ ,

$$\mathbf{n} \cdot (\mathbf{v} - c\mathbf{n}) = \mathbf{n} \cdot \mathbf{v} - c\mathbf{n} \cdot \mathbf{n} = 0 \quad \rightarrow \quad c = \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}$$

therefore

$$\mathbf{p} = \mathbf{v} - c\mathbf{n} = \mathbf{v} - \left( \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n}.$$