Computer Graphics

Splines

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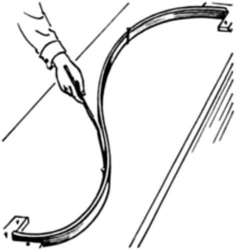

Forms of a 2D Line

What are the "free parameters"?

- slope & y-intercept:
  Slope-intercept form $y = mx + b$
- slope & one point on the line:
  Point-slope form $y - y_1 = m(x - x_1)$
- x- & y-intercepts:
  Intercept form $\frac{x}{a} + \frac{y}{b} = 1$
- Two points on the line:
  Two-point form $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
- Parametric form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} t + \begin{bmatrix} c \\ d \end{bmatrix}$
- General form $ax + by + c = 0 \rightarrow$ What do the free parameters mean?
- Normal form $\mathbf{n} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = d, \quad |\mathbf{n}| = 1$

$\rightarrow$ Best form for design process?

Requirements of Curve Form in Design Process

- Can be modified with intuitive (geometric) free parameters
- Invariant under transformations
  $\rightarrow$ What kind of transformations to be allowed?
- Rendered easily
  $\rightarrow$ Implicit or parametric?

Vectors and Points in Affine Space

- Affine space = vector space + points
- Definition & difference
- Operations
  - addition, subtraction, scalar multiplication, (dot & cross) products
  - "point+vector", "point-vector"
- Linear combinations of vectors: "$\mathbf{v} = \sum_j a_j \mathbf{u}_j$"
- Affine combinations of vectors & points: + "$\sum_j a_j = 1$"
  - Affine combination of points
    $\sum_j a_j \mathbf{p}_j = \mathbf{q} - \mathbf{q} + \sum_j a_j \mathbf{p}_j = \mathbf{q} + \sum_j a_j (\mathbf{p}_j - \mathbf{q})$
    $\rightarrow$ point + sum of vectors
- Convex combinations of vectors & points: + "$\forall a_j \geq 0$"

Homogeneous Representations

- What do we need to represent **any** 3D vector uniquely?
  $\rightarrow$ Three **linearly independent** vectors
- What do we need to represent **any** 3D point uniquely?
  $\rightarrow$ Three linearly independent vectors + fixed point (origin)
- Homogeneous representation
  - Vectors $\mathbf{v} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \phi \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = v_x\mathbf{x} + v_y\mathbf{y} + v_z\mathbf{z}$
  - Points $\mathbf{p} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \phi \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix} = v_x\mathbf{x} + v_y\mathbf{y} + v_z\mathbf{z} + \phi$
  - Validity of operations on vectors & points can be easily checked.

Affine Transformations

- Scaling, rotation, shear, translation, etc.
- Linear transformation $\mathbf{L}$ followed by a translation by $\mathbf{b}$:
  $\mathbf{y} = \mathbf{L}\mathbf{x} + \mathbf{b}$
- In n-D, can be represented by a $(n+1) \times (n+1)$ matrix of the form

$$\begin{bmatrix} \mathbf{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{L} & \mathbf{b} \\ 0 \cdots 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

  - How about vectors?
- Affine combination of points is invariant under affine transformations:
  For $\sum_j a_j = 1$, $\mathbf{A}(\sum_j a_j \mathbf{p}_j) = \sum_j a_j (\mathbf{A}\mathbf{p}_j)$
  $\rightarrow$ Why is this important?

Representation of a Line Segment

- By the previous arguments...
  If a curve is defined by a affine combination of (a finite number of) points ("control points"), any affine transformation of the curve can be achieved by applying the affine transformation only to the control points.
- Representation of a line segment

$$c(t) = (1-t)\mathbf{p} + t\mathbf{q}$$

  - Parametric representation
  - Affine combination of two points $\mathbf{p}$ and $\mathbf{q}$
    $\rightarrow$ $\mathbf{p}$ and $\mathbf{q}$ are the control points
  - A line segment connecting $\mathbf{p}$ and $\mathbf{q}$ for $0 \leq t \leq 1$:
    $c(0) = \mathbf{p}$ and $c(1) = \mathbf{q}$

Curves for Design Process

- What kind of functions to use?
- What are the control points?
- How can we represent a polynomial curve as an affine combination of the control points?

Bézier Curves

$$\mathbf{b}(t) = \sum_{j=0}^{n} \beta_j^n(t)\mathbf{p}_j$$

- Polynomial curve of degree n
- Parametric representation (usually defined for $0 \leq t \leq 1$)
- Represented as an affine combination of control points ($\{\mathbf{p}_j\}_{j=0}^n$) where the coefficients are the Bernstein basis polynomials defined as

$$\beta_j^n(t) := \binom{n}{j} t^j (1-t)^{n-j}$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ is the binomial coefficient.

Properties of Bernstein Basis Polynomials

- Non-negativity: $\beta_j^n(t) \geq 0$ for $0 \leq t \leq 1$
- Partition of unity: $\sum_{j=0}^{n} \beta_j^n(t) = 1$
- $\beta_j^n(0) = \delta_{j,0}$ and $\beta_j^n(1) = \delta_{j,n}$ where $\delta_{j,k} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ is the Kronecker delta function.
- Symmetry: $\beta_j^n(1-t) = \beta_{n-j}^n(t)$
- Recurrence: $\beta_0^0(t) \equiv 1$ and $\beta_j^n(t) = (1-t)\beta_j^{n-1}(t) + t\beta_{j-1}^{n-1}(t)$
- Derivative: $\frac{d\beta_j^n}{dt}(t) = n(\beta_{j-1}^{n-1}(t) - \beta_j^{n-1}(t))$
- If $n \neq 0$, then $\beta_j^n(t)$ has a unique local maximum on the interval $[0,1]$ at $t = j/n$.
- $\{\beta_j^n\}_{j=0}^{n}$ form a basis of the vector space of polynomials of degree n.
- Degree elevation:
  $\beta_j^{n-1}(t) = \frac{1}{n}\Big((n-j)\beta_j^n(t) + (j+1)\beta_{j+1}^n(t)\Big)$

Properties of Bézier Curves

- For $0 \le t \le 1$, all the points on $\mathbf{b}(t)$ are the convex combinations of the control points.
- Convexity: For $0 \le t \le 1$, $\mathbf{b}(t)$ lies inside the convex hull of the control points.
- Endpoint interpolation: $\mathbf{b}(0) = \mathbf{p}_0$ and $\mathbf{b}(1) = \mathbf{p}_n$
- Symmetry
- Recurrence $\rightarrow$ Can be evaluated in numerically stable way by the de Casteljau's algorithm
- The effect of the control point is largest near that point.
- Given a Bézier curve, its control points are unique.
- Subdivision: $\mathbf{b}(ct) = \sum_{j=0}^{n} \beta_j^n(t) \left(\sum_{k=0}^{j} \beta_k^j(c)\mathbf{p}_k \right)$
- Degree elevation: $\mathbf{b}(t) = \sum_{j=0}^{n} \beta_j^n(t)\mathbf{p}_j = \sum_{j=0}^{n+1} \beta_j^{n+1}(t)\mathbf{q}_j$ where $\mathbf{q}_j = \frac{j}{n+1}\mathbf{p}_{j-1} + \left(1 - \frac{j}{n+1}\right)\mathbf{p}_j$

Composite Curves

- How to make curve pieces connected smoothly?

Polynomial Interpolation

- For given points $\{\mathbf{p}_i\}_{i=0}^{n}$, is there a polynomial that *interpolates* all the points?
- If it exists, is it unique?
- What is its degree?
- How can we find it?
  - $\mathbf{p}(t) = \sum_{j=0}^{n} \mathbf{a}_j t^j$
  - $\mathbf{p}(t_i) = \mathbf{p}_i = \sum_{j=0}^{n} \mathbf{a}_j t_i^j, \quad i = 0, \dots, n$
  - $[\mathbf{p}_i] = \left[t_i^j\right][\mathbf{a}_j] \rightarrow [\mathbf{a}_j] = \left[t_i^j\right]^{-1}[\mathbf{p}_i]$
  - $\det\left[t_i^j\right]$: *Vandermonde polynomial* or *Vandermonde determinant*
- Better methods?

Aitken's algorithm

- To find an interpolating polynomial
- Recursive algorithm

$$\mathbf{p}_i^r(t) = \frac{t_{i+r} - t}{t_{i+r} - t_i}\mathbf{p}_i^{r-1}(t) + \frac{t - t_i}{t_{i+r} - t_i}\mathbf{p}_{i+1}^{r-1}(t), \quad \begin{cases} r = 1, \ldots, n \\ i = 0, \ldots, n - r \end{cases}$$

  where $\mathbf{p}_i^0(t) := \mathbf{p}_i$
- Properties
  - Affine invariance? $\rightarrow$ Yes
  - Linear precision? $\rightarrow$ Yes
  - Convex hull property? $\rightarrow$ No
  - Variation diminishing? $\rightarrow$ No

Lagrange Polynomials

- Defined as

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{p}_i L_i^n(t)$$

where

$$L_i^n(t) = \frac{\prod_{j=0, j\neq i}^{n} (t - t_j)}{\prod_{j=0, j\neq i}^{n} (t_i - t_j)}$$

$\rightarrow L_i^n(t_j) = \delta_{i,j}$

Limits of Interpolating Polynomials

- Runge's phenomenon
- Evaluation cost

Cubic Hermite Interpolation

- Interpolates two points and the tangent vectors at each:

$$\mathbf{p}(0) = \mathbf{p}_0, \dot{\mathbf{p}}(0) = \mathbf{m}_0, \dot{\mathbf{p}}(1) = \mathbf{m}_1, \mathbf{p}(1) = \mathbf{p}_1$$

- In Bézier form,

$$\mathbf{p}(t) = \mathbf{p}_0\beta_0^3(t) + \left(\mathbf{p}_0 + \frac{1}{3}\mathbf{m}_0\right)\beta_1^3(t) + \left(\mathbf{p}_1 - \frac{1}{3}\mathbf{m}_1\right)\beta_2^3(t) + \mathbf{p}_1\beta_3^3(t)$$

- In cardinal form,

$$\mathbf{p}(t) = \mathbf{p}_0 H_0^3(t) + \mathbf{m}_0 H_1^3(t) + \mathbf{m}_1 H_2^3(t) + \mathbf{p}_1 H_3^3(t)$$

$\rightarrow H_j^3(t) =$?

- Properties
  - Cardinality
  - Affine invariance $H_0^3(t) + H_3^3(t) \equiv 1$
  - Not invariant under affine *domain* transformations
  - Not symmetric

B-Splines: Motivation

We want...

- A polynomial curve
- Defined as affine combinations of finite number of control points
- Not necessarily interpolates control points, but approximates them
- Low degree even when the number of control points grows
- True local control
- No worry for connecting smoothly
- Stable evaluation

B-Splines: Definition

With the *knot sqeuence* of size m

$$t_0 \leq t_1 \leq \cdots \leq t_{m-1},$$

a B-spline of degree n with control points $\{\mathbf{p}_0, \cdots, \mathbf{p}_{m-n-2}\}$ is defined as

$$\mathbf{s}(t) = \sum_{j=0}^{m-n-2} \mathbf{p}_j B_j^n(t), \quad t \in [t_n, t_{m-n-1}]$$

where the basis is defined recursively as

$$B_j^0(t) := \begin{cases} 1 & \text{if } t_j \leq t \leq t_{j+1} \\ 0 & \text{otherwise} \end{cases}, \quad j = 0, \ldots, m-2$$

$$B_j^n(t) := \frac{t - t_j}{t_{j+n} - t_j} B_j^{n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} B_{j+1}^{n-1}(t), \quad j = 0, \ldots, m-n-2$$

- Examples