Mathematical Models for Engineering Problems and Differential Equations

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Practical Methods

What if a solution cannot be expressed in terms of elementry functions?

- Graphical methods (“direction field” in Chap.1)
- Numerical methods (Chap.10)
- Series methods (Chap.9)

Outline

Power Series

Definition

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{k=0}^{\infty} a_k(x - x_0)^k,$$

where $a_0, a_1, \cdots, x_0$ are constants.

A power series may...

1. Converge only for the single value $x = x_0$.
2. Converge *absolutely* for values of x in a neighborhood of x_0, i.e., converge for $|x - x_0| < h$; diverge for $|x - x_0| > h$. At the end points $x_0 \pm h$, it may either converge or diverge.
3. Converge absolutely for all values of x, i.e., for $-\infty < x < \infty$.

Power Series (cont'd)

Interval of convergence

- The set of values of x for which the power series converges.
- To determine an interval of convergence, use the following "ratio test":
  A series
  $$u_1 + u_2 + \cdots + u_n + \cdots = \sum_{k=1}^{\infty} u_k$$
  converges *absolutely* if
  $$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = k < 1.$$
  $\rightarrow$ Find the values of x which satisfy the above inequality.

Power Series (cont'd)

Theorem 37.16

If a power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

converges on an interval $I : |x - x_0| < R$, where R is a positive constant, then the power series **defines a function** $f(x)$ which is continuous for each x in I.

- If a power series converges, which function does it define? $\rightarrow$ (mostly) hard to answer
- Conversely, given a continuous function (on an interval I), is there a power series which defines it?

Power Series (cont'd)

- Theorem 37.2: If

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \quad I : |x - x_0| < R,$$

  then

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}, \quad I : |x - x_0| < R.$$

- Theorem 37.23: If

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k, \quad I : |x - x_0| < R,$$

  then $f(x) = g(x)$ iff $a_k = b_k, \forall k$.

- Theorem 37.24: If

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \quad I : |x - x_0| < R,$$

  then $a_k = \dfrac{f^k(x_0)}{k!}, \quad \forall k.$

Taylor Series Expansion

Definition

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \quad |x - x_0| < R.$$

- Maclaurin series: Taylor series with $x_0 = 0$.
- Is the Taylor series expansion of a function $f(x)$ actually the power series that defines (or converges to) the given function $f(x)$? (e.g., $f(x) = e^{-1/x^2}$)
  $\rightarrow$ Taylor's theorem required.

Taylor's Theorem

Let (Taylor series with remainder)

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x), \quad |x - x_0| < h,$$

where

$$R_n(x) := \sum_{k=n+1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \frac{f^{(n+1)}(X)(x - x_0)^{n+1}}{(n+1)!}, \quad \text{(Lagrange form)}$$

and X is between x and x_0.
The Taylor series $\sum_{k=0}^{\infty} f^{(k)}(x_0)(x - x_0)^k$ defines (or converges to) $f(x)$ on $I : |x - x_0| < h$ iff $\lim_{n\to\infty} R_x(x) \to 0$.

- Analytic function

Solution of L.D.E.s by Series Methods

To solve a L.D.E. of the form

$$y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x).$$

Theorem 37.51

If each of $f_0(x), \cdots, f_{n-1}(x), Q(x)$ is analytic at $x = x_0$, then there is a unique solution $y(x)$ of the D.E. which is also analytic at $x = x_0$ satisfying the n initial conditions

$$y(x_0) = a_0, y'(x_0) = a_1, \cdots, y^{(n-1)}(x_0) = a_{n-1}.$$

$\rightarrow$ But how to find the solution?

1. Successive differentiations method
2. Undetermined coefficients method

Method #1. Successive Differentiation

- Find
$$\{f^{(k)}(x_0)\}_{k=0}^{\infty}$$
by successively differentiating the original D.E. and evaluating at $x = x_0$ using the initial conditions.
- Usually find only several terms at the beginning.
$\rightarrow$ Approximate solution

Method #2. Undetermined Coefficients

- No differentiating required $\rightarrow$ Useful when it is difficult to obtain successive derivatives.
- Set

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

  and substitute into the D.E. to find $\{a_k\}_{k=0}^{\infty}$.
- Usually find only several terms at the beginning.
  $\rightarrow$ Approximate solution

Outline

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Chapter 9: Series Methods

Outline

Ordinary and Singular Points

Can we still apply the methods in Lesson 37 when Theorem 37.51 (p.538) is not satisfied?

Example: Solve the followwing at $x = 0$.

$$x^2y'' + xy' + (x^2 - 1/4)y = 0$$

Definition

For the D.E.

$$y^{(n)} + F_{n-1}(x)y^{(n-1)} + \cdots + F_1(x)y' + F_0(x)y = Q(x),$$

a point $x = x_0$ is called

- an *ordinary point* of the D.E. if *each* function $F_0, F_1, \cdots, F_{n-1}$ and Q is analytic at $x = x_0$ and
- a *singular point* of the D.E. if *one of more* of the functions $F_0, F_1, \cdots, F_{n-1}$ and Q is *not* analytic at $x = x_0$.

Regular and Irregular Singularities

For a second order L.D.E.

$$y'' + F_1(x)y' + F_2(x)y = 0,$$

where F_1 and F_2 are continuous function of x on a common interval I, a point $x = x_0$ is called

- a *regular singularity* of the D.E. if $(x - x_0)F_1(x)$ and $(x - x_0)^2F_2(x)$ are both analytic at $x = x_0$ and
  $\rightarrow$ can be solved by "method of Frobenius series."
- an *irregular singularity* of the D.E. either $(x - x_0)F_1(x)$ or $(x - x_0)^2F_2(x)$ is *not* analytic at $x = x_0$.
  $\rightarrow$ too difficult!

Method of Frobenius Series

Frobenius series

$$y = (x - x_0)^m \sum_{j=0}^{\infty} a_j (x - x_0)^j, \quad a_0 \neq 0.$$

Note: Taylor series is a special case of a Frobenius series. (when m is a non-negative integer)

Theorem 40.32 (p.573)

If $x = x_0$ is a regular singularity of the D.E.

$$(x - x_0)^2 y'' + (x - x_0) f_x(x) y' + f_2(x) y = 0,$$

where $f_1(x) := (x - x_0)F_1(x)$ and $f_2(x) := (x - x_0)^2 F_2(x)$, if each Taylor series expansion of $f_1(x)$ and $f_2(x)$ is valid in the interval $I : |x - x_0| < r$, then at least one Frobenius series solution is also valid in I except perhaps for $x = x_0$.

Method of Frobenius Series (cont'd)

1. Assume $x_0 = 0$ then the D.E. becomes

$$x^2y'' + xf_1(x)y' + f_2(x)y = 0.$$

2. Since $f_1(x)$ and $f_2(x)$ are analytic at $x = 0$, $f_1(x) = \sum_{j=0}^{\infty} b_j x^j$ and $f_2(x) = \sum_{j=0}^{\infty} c_j x^j$.
3. Substitute a Frobenius series $y(x) = x^m \sum_{j=0}^{\infty} a_j x^j$ and its derivatives, $y'(x)$ and $y''(x)$, into the D.E.
4. After arrangement, we get (40.37) (p.574)

$$a_0[m(m-1) + b_0 m + c_0]x^m + (\cdots)x^{m+1} + \cdots + (\cdots)x^{m+n} \equiv 0.$$

5. Indical equation: $m(m-1) + b_0 m + c_0 = 0$.
   - Case 1: m_1 and m_2 are distinct and their difference is not an integer.
   - Caes 2: m_1 and m_2 differ by an integer N.
     - Case 2A: The coefficient of $a_N = 0$ and the remainder terms in the coefficients of x^{m+N} also add to zero.
     - Case 2B: The coefficient of $a_N = 0$ but the remainder terms in the coefficients of x^{m+N} do not add to zero.
   - Case 3: $m_1 = m_2$.

Case 1: m_1 and m_2 are distinct and their difference is not an integer

1. For $m_1, j = 1, 2$, find a set of $a_1, a_2, \cdots, a_n, \cdots$ in terms of m_1 and a_0 which results in a solution y_1.
2. Do the same for m_2 and find another solution y_2.
3. Due to Theorem 40.39, the general solution is $y = c_1y_1 + c_2y_2$.

Caes 2: m_1 and m_2 differ by an integer N.

1. Let the roots are m and $m+N$, $N > 0$.
2. For $m+N$, the indical equation becomes

$$(m+N)(m+N-1) + b_0(m+N) + c_0 = 0.$$

3. In (40.37), this is the same as the first term of the coefficient of x^{m+N}.
   $\rightarrow$ Two sub-cases.

Case 2A: The coefficient of $a_N = 0$ and the remainder terms in the coefficients of x^{m+N} also add to zero.

1. In this case, the coefficient of x^{m+N} in (40.37) is zeo regardless of a_N.
2. For m, we can find two sets of coefficients $a_1, a_2, \cdots$, thus two solutions y_1 and y_2, one in terms of (arbitrary) a_0 and the other in terms of (arbitrary) a_N.
3. The solution, say y_3, obtained by $m + N$ in terms of a_0 is linearly dependent.
4. Therefore the general solution is $y = y_1 + y_2$. (Note that y_1 and y_2 each contain arbitrary constants a_0 and a_N.)

Case 2B: The coefficient of $a_N = 0$ but the remainder terms in the coefficients of x^{m+N} do not add to zero.

1. There is only one Frobenius series solution $y_1(x)$, obtained for $m + N$ in terms of a_0.
2. A second independent solution has the form

$$y_2(x) = u(x) - b_N y_1(x) \log x, \quad x > 0,$$

where

$$u(x) := x^m \sum_{j=0}^{\infty} b_j x^j$$

is a Frobenius solution.

Case 3: $m_1 = m_2$.

1. There is only one Frobenius series solution in terms of a_0.
2. A second solution can be obtained by the method of Case 2B with $N = 0$.

Outline

Legendre Equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}y(x)\right] + k(k+1)y = (1-x^2)y'' - 2xy' + k(k+1)y = 0$$

- $x = 0$ is an ordinary point
  $\rightarrow$ can be solved by the method in Lesson 37B. ("successive differentiations" or "undetermined coefficients")
- $x = \pm 1$ are regular singular points.
- $$f_0(x) := \frac{k(k+1)}{1-x^2} \quad \text{and} \quad f_1(x) := -\frac{2x}{1-x^2}$$
  are both valid for $|x| < 1$.
  $\rightarrow$ has a series solution of x valid for $|x| < 1$.
- General solution:

  $$y(x) = c_1 P_k(x) + c_2 Q_k(x),$$

  where $P_k(x)$ is called "Legendre polynomial" and $Q_k(x)$ is called "Legendre function of the second kind." (Mathworld)

Applications of Legendre Equation

- Steady state temperature within a solid spherical ball when the temparature at points of its boundary is know.
  $\rightarrow$ Solving Laplace's equation in spherical coodrinates. (Mathworld)
- Quantum mechanical model of the hydrogen atom

Outline

Chapter 9: Series Methods

Lesson 37: Power Series Solutions of Linear Differential Equations.

Lesson 38. Series Solution of $y' = f(x, y)$

Lesson 39. Series Solution of a Nonlinear Differential Equation of Ord

Lesson 40. Ordinary Points and Singularities of a Linear Differential E

Lesson 41. The Legendre Differential Equation. Legendre Functions.

Lesson 42. The Bessel Differential Equation. Bessel Function of the F

Lesson 43. The Laguerre Differential Equation. Laguerre Polynomials

Bessel Equation

$$x^2y'' + xy' + (x^2 - k^2)y = 0$$

- $x = 0$ is a regular singularity.
  $\longrightarrow$ Frobenius solution

$$y(x) = x^m \sum_{j=0}^{\infty} a_k x^k.$$

- Roots of the indical equation: $\pm k$.
- General solution:

$$y(x) = \begin{cases} c_1 J_k(x) + c_2 J_{-k}(x) & k \neq 0, 1, 2, 3, \cdots \\ c_1 J_k(x) + c_2 N_k(x) & k = 1, 2, 3, \cdots \end{cases}$$

where $J_k(x)$ is called "Bessel function of the first kind of index k" and $N_k(x)$ is called "Bessel function of the second kind of index k." (See p.584 for $k = 0$.)

Applications of Bessel Eqauation

- Lapalce's equation in cylindrical coordinates.
- ...and more (Wikipedia)

Outline

Laguerre Equation

$$xy'' + (1 - x)y' + ky = 0, \quad k \text{ real}$$

- Has non-singular solutions only if k is an integer.
- $x = 0$ is a regular singularity
  $\rightarrow$ Frobenius solution

$$y(x) = x^m \sum_{j=0}^{\infty} a_k x^k.$$

- Roots of the indical equation: multiple 0.
  $\rightarrow$ only one Frobenius solution and the second solution has a logarithmic form. (40.51)
- Laguerre polynomial $L_k(x)$ is a solution. (when $k = 0, 1, 2, \cdots$.)

Applications of Laguerre Equation

- Particle in a spherically symmetric potential
  $\rightarrow$ Radial equation of the Schrödinger equation
- Laguerre polynomials are used for Gaussian quadrature to numerically compute integrals of the form $\int_0^\infty f(x)dx$.