Mathematical Models for Engineering Problems and Differential Equations

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Table of contents

Chapter 6: Problems Leading to Linear Differential Equations of Order Tw

L.D.E. of This Chapter

The motion of a particle whose equation of motion satisfiesa D.E. of the form

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = f(t).$$

- $r = 0$ and $f(t) \equiv 0 \rightarrow \frac{d^2x}{dt^2} + \omega_0^2 x = 0$
  $\rightarrow$ 28A: Free undamped motion (simple harmonic motion)
- $r = 0$ and $f(t) \not\equiv 0 \rightarrow \frac{d^2x}{dt^2} + \omega_0^2 x = f(t)$
  $\rightarrow$ 28D: Forced undamped motion
- $r \neq 0$ and $f(t) \equiv 0 \rightarrow \frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = 0$
  $\rightarrow$ 29A: Free damped motion (damped harmonic motion)
- $r \neq 0$ and $f(t) \not\equiv 0 \rightarrow \frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = f(t)$
  $\rightarrow$ 29B: Forced damped motion

$\rightarrow$ Summarized in the table on p.365.

Lesson 28: Undamped Motion.

Free undamped motion (simple harmonic motion)

A particle will be said to execute “simple harmonic motion” if its equation of motion satisfies

$$\frac{d^2x}{dx^2} + \omega_0^2 x = 0$$

where ω_0 is a positive constant and $x(t)$ is the positon of the particle.

- The motion of a particle oscillating back and forth about a fixed point of equilibrium.
- Examples: displaced helical spring, a pendulum

Free undamped motion (simple harmonic motion) (cont'd)

The solution is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

or

$$x(t) = \sqrt{c_1^2 + c_2^2} \sin(\omega_0 t + \delta) = c \sin(\omega_0 t + \delta)$$

or

$$x(t) = \sqrt{c_1^2 + c_2^2} \cos(\omega_0 t + \delta) = c \cos(\omega_0 t + \delta).$$

Free undamped motion (simple harmonic motion) (cont'd)

Example 28.15 (p.314)

Description of the motion:

- maximum displacement?
- relation of position and velocity?
- maximum speed?

Free undamped motion (simple harmonic motion) (cont'd)

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$
$$x(t) = \sqrt{c_1^2 + c_2^2} \sin(\omega_0 t + \delta) = c \sin(\omega_0 t + \delta)$$
$$x(t) = \sqrt{c_1^2 + c_2^2} \cos(\omega_0 t + \delta) = c \cos(\omega_0 t + \delta).$$

- equilibrium position
- amplitude (c)
- phase angle (δ)
- period ($T = 2\pi/\omega_0$)
- natural (undamped) frequency (cycles per second $\nu = 1/T = \omega_0/2\pi$)
- natural (undamped) frequency (radians per second ω_0)

Harmonic Oscillators

A A particle attached to an elastic helical spring: “Hook’s law”

B A simple pendulum

Example A: A particle attached to an elastic helical spring

Hook's law

$$\mathbf{F} = -k\mathbf{x}$$

- $\mathbf{x}$ displacement of the end of the spring from its equilibrium position
- $\mathbf{F}$ restoring force exerted by the spring
- k force/spring/stiffness constant

Example A: A particle attached to an elastic helical spring (cont'd)

Fig.28.6 (p.324)

- What's the upward force of the spring?
- What's the downward force due to the weight of the particle?
- What's the relation of the forces in equilibrium?
- What's the D.E. of the position of the particle (mass m) when stretched by y?
- What is the solution?

Example B: A simple pendulum

Fig.28.7 (p.327)

- θ angle of swing
- w angular velocity
- F effective force (which moves the pendulum) due to the weight of the pendulum
- D.E. of $\theta(t)$?
- Is it the D.E. of simple harmonic motion?
- Which assumption do we need?

Forced Undamped Motion

$$m\frac{d^2y}{dt^2} + m\omega_0^2 y = f(t) \quad \rightarrow \quad \frac{d^2y}{dt^2} + \omega_0^2 y = \frac{1}{m}f(t)$$

- $f(t)$ forcing function attached to the system
- With $f(t) = mF\sin(\omega t + \beta)$, we consider two cases: (i) $\omega \neq \omega_0$ and (ii) $\omega = \omega_0$.
- ω is called ‘impressed frequency’ or ‘forcing frequency’.
- The complementary function:

$$y_c = c\sin(\omega_0 t + \delta).$$

Case 1: $\omega \neq \omega_0$

- General solution:

$$y = c\sin(\omega_0 t + \delta) + \frac{F}{\omega_0^2 - \omega^2}\sin(\omega t + \beta).$$

  $\rightarrow$ sum of two simple harmonic motions.
- Stable motion (see Fig.28.85 on p.340)
- What if $\omega \approx \omega_0$?

Case 2: $\omega = \omega_0$

- Which method to use?
- General solution:

$$y = c \sin(\omega_0 t + \delta) - \frac{F}{2\omega_0} t \cos(\omega_0 t + \beta).$$

- Unstable motion (see Fig.28.94 on p.341)
  $\rightarrow$ "undamped resonance", "undamped resonant frequency"
- See "mechanical resonance" at Wikipedia.

Lesson 29: Damped Motion.

Damped Harmonic Motion

Definition

$$m\frac{d^2y}{dt^2} + 2mr\frac{dy}{dt} + m\omega_0^2 y = 0, \qquad \frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + \omega_0^2 y = 0$$

- $2mr > 0$: coefficient of resistance
- Can be solved via “characteristic equation” (Lesson 20)

$$m^2 + 2rm + \omega_0^2 = 0 \rightarrow m = -r \pm \sqrt{r^2 - \omega_0^2}$$

- Three cases
  1. $r^2 > \omega_0^2$
  2. $r^2 = \omega_0^2$
  3. $r^2 < \omega_0^2$

Case 1. $r^2 > \omega_0^2$

- Solution

$$y = c_1 e^{At} + c_2 e^{Bt},$$

  where

$$A := -r + \sqrt{r^2 - \omega_0^2} < 0 \quad \text{and} \quad B := -r - \sqrt{r^2 - \omega_0^2} < 0.$$

- Two cases (assuming $c_1 \neq 0$ and $c_2 \neq 0$)
  1. $c_1 c_2 > 0$: y never crosses the t axis.
  2. $c_1 c_2 < 0$: y crosses the t axis only once.
- $\lim_{t \to \infty} y = 0 \to$ Dies out with time.
- Derivative

$$\frac{dy}{dt} = c_1 A e^{At} + c_2 B e^{Bt}$$

  $\to$ At most one value of t such that $dy/dt = 0$

  $\to$ At most only one extreme point

  $\to$ Non-oscillatory
- "overdamped":

  resisting (damping) force > restoring force
- Figure 29.17 (p.349)

Case 2. $r^2 = \omega_0^2$

- Solution
$$y = c_1 e^{-rt} + c_2 t e^{-rt}$$
- $\lim_{t\to\infty} y = 0 \longrightarrow$ Dies out with time
- Non-oscillatory
- "critically damped"
resisting (damping) force = restoring force

Case 3. $r^2 < \omega_0^2$

- Solution (by Lesson 20D)

$$y = ce^{-rt}\sin\left(\sqrt{\omega_0^2 - r^2}\,t + \delta\right)$$

- Oscillatory (damped periodic)
- $\lim_{t\to\infty} y = 0$ due to the "damping factor" e^{-rt}
- $\tau := \arg_t(e^{-rt} = 1/e) = 1/r$
  "time constant"
- "underdamped"
  resisting (damping) force $<$ restoring force
- Figure 29.33 (p.351)

Forced Motion with Damping

Definition

$$m\frac{d^2y}{dt^2} + 2mr\frac{dy}{dt} + m\omega_0^2 y = f(t), \qquad \frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + \omega_0^2 y = \frac{1}{m}f(t),$$

- Consider

$$f(t) = mF\sin(\omega t + \beta)$$

- Particular solution (Lesson 21A, case 1)

$$y_p = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2r\omega)^2}}\sin(\omega t + \beta - \alpha),$$

where

$$\cos\alpha = \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (wr\omega)^2}, \quad \text{and} \quad \sin\alpha = \frac{2r\omega}{(\omega_0^2 - \omega^2)^2 + (wr\omega)^2}.$$

Forced Motion with Damping (cont'd)

- General solution

$$y = y_c + \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2r\omega)^2}} \sin(\omega t + \beta - \alpha),$$

  - y_c dies out with time $\rightarrow$ "transient motion"
  - $\lim_{t\to\infty} y = y_p \rightarrow$ "steady state motion"
  - Steady state motion has the same frequency as $f(t)$, but shifted.
  - If $\omega = \omega_0$, the amplitude $A(\omega = \omega_0) = F/2r\omega_0$.
  - If $\omega \neq \omega_0$, then

$$A_{\max} := \max_{\omega} A(\omega) = A\left(\omega = \sqrt{\omega_0^2 - 2r^2}\right) = \frac{F}{2r\sqrt{\omega_0^2 - r^2}}.$$

    $\rightarrow$ "resonant system"
  - If $\omega \neq \omega_0$ and r is small (small "resistance") the amplitude

$$A_{\max} \approx A(\omega = \omega_0) = \frac{F}{2r\omega_0}$$

    gets very large!

Lesson 30: Electric Circuits. Analog Computation.

Lesson 30M: Miscellaneous Types of Problems Leading to Linear Equatio