

Mathematical Models for Engineering Problems and Differential Equations

Minho Kim

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Chapter 5: Operators and Laplace Transform

Lesson 24: Differential and Polynomial Operators.

Operator

Definition

A mathematical device which converts one function into another.

Examples:

- ▶ differentiating operator: $f(x) \rightarrow f'(x)$
- ▶ integrating operator: $f(x) \rightarrow F(x) = \int_{x_0}^x f(t)dt$

Differential Operator

$$D^n y = y^{(n)}$$

Polynomial Operator

- ▶ A linear combination of differential operators of orders 0 to n

$$P(D) = a_0 + a_1D + a_2D^2 + \cdots + a_nD^n, \quad a_n \neq 0$$

- ▶ $P(D)y := a_ny^{(n)} + \cdots + a_1y' + a_0y$, $a_n \neq 0$ A linear differential equation with constant coefficients

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = Q(x), \quad a_n \neq 0$$

can be expressed as

$$P(D)y = Q(x).$$

- ▶ Linear: $P(D)(c_1y_1 + c_2y_2) = c_1P(D)y_1 + c_2P(D)y_2$
 1. If $P(D)y_1 = 0, \dots, P(D)y_n = 0$ then $P(D)(c_1y_1 + \cdots + c_ny_n) = 0$.
 2. If $P(D)y_c = 0$ and $P(D)y_p = Q(x)$ then $P(D)(y_c + y_p) = Q(x)$.

Polynomial Operator (cont'd)

- ▶ Principle of superposition:

If $P(D)y_{1p} = Q_1(x)$, $P(D)y_{2p} = Q_2(x)$, \dots , $P(D)y_{np} = Q_n(x)$, then
 $P(D)(y_{1p} + y_{2p} + \dots + y_{np}) = Q_1(x) + Q_2(x) + \dots + Q_n(x)$.

Algebraic Properties of Polynomial Operators

- ▶ $(P_1 + P_2)(D)y := P_1(D)y + P_2(D)y$
- ▶ $[h(x)P(D)]y := h(x)[P(D)y]$
- ▶ $[P_1(D)P_2(D)]y := P_1(D)[P_2(D)y]$
- ▶ Commutitative:

$$[P_1(D)P_2(D)]y = [P_2(D)P_1(D)]y$$

- ▶ Associative:

$$P_1(D)[P_2(D)P_3(D)] = [P_1(D)P_2(D)]P_3(D) = P_1(D)P_2(D)P_3(D)$$

- ▶ Distributive:

$$P_1(D)[P_2(D) + P_3(D)] = P_1(D)P_2(D) + P_1(D)P_3(D)$$

Factoring of Polynomial Operators

If

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0, \quad a_n \neq 0,$$

where a_0, a_1, \cdots, a_n are constants, then

$$P(D) = a_n (D - r_1)(D - r_2) \cdots (D - r_n),$$

where r_1, r_2, \cdots, r_n are the real or imaginary roots of the characteristic equation of $P(D)y = 0$.

Exponential Shift Theorem for Polynomial Operators

Theorem

If

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0, \quad a_n \neq 0$$

then

$$P(D)(u(x)e^{ax}) = e^{ax}P(D+a)u(x).$$

Corollary

$$(D - a)^n(u(x)e^{ax}) = e^{ax}D^n u(x)$$

Corollary

$$P(D)(ce^{ax}) = ce^{ax}P(a)$$

Solving a Linear D.E. by Means of Polynomial Operators

1. Factor $P(D)$ then the D.E. is expressed as

$$P(D)y = (D - r_1)(D - r_2) \cdots (D - r_n)y = Q(x)$$

2. Let $u_1 = (D - r_2) \cdots (D - r_n)y$ then the D.E. becomes

$$(D - r_1)u_1 = Q(x).$$

3. Find u_1 then we get a D.E.

$$(D - r_2)(D - r_3) \cdots (D - r_n)y = u_1(x).$$

4. Let $u_2 = (D - r_3) \cdots (D - r_n)y$ then the D.E. becomes

$$(D - r_2)u_2 = u_1(x).$$

5. Repeat.

Lesson 25: Inverse Operators.

Inverse Operators

Definition

Let $P(D)y = Q(x)$ where $Q(x)$ is the special function consisting only of such terms as b , x^k , $\sin ax$, $\cos ax$, and a finite number of combinations of such terms. (a , b constants and k a positive integer) Then the *inverse operator* of $P(D)$, written as $P^{-1}(D)$ or $1/P(D)$, is defined as an operator which, when operating on $Q(x)$, will give the particular solution y_p of the D.E. that contains no constant multiples of a term in the complementary function y_c , i.e.,

$$P^{-1}(D)Q(x) = y_p \quad \text{or} \quad \frac{1}{P(D)}Q(x) = y_p.$$

- ▶ What does “ $D^{-n}Q(x)$ ” mean?
- ▶ $P^{-1}(D)(0) = ?$
- ▶ $P(D)[P^{-1}(D)Q] = ?$

Solving “ $P(D)y = Q(x)$ ” by Means of Inverse Operators

1. $Q(x) = bx^k$ and $P(D) = D - a_0$
2. $Q(x) = bx^k$ and $P(D) = a_n D^n + \cdots + a_1 D_1$
3. $Q(x) = be^{ax}$ and $P(a) \neq 0$
4. $Q(x) = b \sin ax$ or $b \cos ax$
5. $Q(x) = u(x)e^{ax}$, $u(x)$ is a polynomial in x
6. $Q(x) = be^{ax}$ and $P(a) = 0$
7. $Q(x) = Q_1(x) + \cdots + Q_n(x)$

1. $Q(x) = bx^k$ and $P(D) = D - a_0$

1. The D.E.:

$$y' - a_0y = bx^k, \quad a_0 \neq 0.$$

2. This is the Case 1. of Lesson 21A; “No term of $Q(x)$ is the same as a term of y_c . In this case, y_p is a linear combination of the terms in $Q(x)$ and all its linearly independent derivatives.”

3. Trial function: $y_p = A_k x^k + A_{k-1} x^{k-1} + \cdots + A_1 x + A_0$.

4. We get

$$y_p = -\frac{b}{a_0} \left(x^k + \frac{k}{a_0} x^{k-1} + \frac{k(k-1)}{a_0^2} x^{k-2} + \cdots + \frac{k!}{a_0^k} \right), \quad a_0 \neq 0.$$

5. The same result can be obtained as follows:

$$\begin{aligned} y_p &= \frac{1}{D - a_0} (bx^k) = \frac{1}{-a_0 \left(1 - \frac{D}{a_0}\right)} (bx^k) \\ &= -\frac{1}{a_0} \left(1 + \frac{D}{a_0} + \frac{D^2}{a_0^2} + \cdots + \frac{D^k}{a_0^k} \right) (bx^k). \end{aligned}$$

$$2. Q(x) = bx^k \text{ and } P(D) = a_n D^n + \cdots + a_1 D_1$$

1. In general, if

$$P(D) = D^r(a_n D^{n-r} + \cdots + a_r).$$

2. By the inverse operator, we can solve as follows:

▶ differentiating followed by integrating:

$$y_p = \frac{1}{D^r} \left[\frac{1}{a_n D^{n-r} + \cdots + a_{r+1} D + a_r} (bx^k) \right], \quad a_r \neq 0.$$

▶ integrating followed by differentiating:

$$y_p = \frac{1}{a_n D^{n-r} + \cdots + a_{r+1} D + a_r} \left[\frac{1}{D^r} (bx^k) \right], \quad a_r \neq 0.$$

→ May introduce terms that are constant multiples of terms in y_c .

2. $Q(x) = bx^k$ and $P(D) = a_n D^n + \cdots + a_1 D_1$: Example

Example 25.36

$$y'' - 2y' = 5, \quad (D^2 - 2D)y = 5.$$

1. First, note that

$$\begin{aligned} \frac{1}{D-2} &= -\frac{1}{2} \left(1 + \frac{D/2}{1-D/2} \right) = -\frac{1}{2} \left(1 + \frac{D}{2} + \frac{D^2/4}{1-D/2} \right) \\ &= -\frac{1}{2} \left(1 + \frac{D}{2} + \frac{D^2}{4} + \frac{D^3}{8} + \cdots + \frac{D^n}{2^n} + \cdots \right) \end{aligned}$$

2. $y_c = c_1 e^{2x} + c_2$.

2. $Q(x) = bx^k$ and $P(D) = a_n D^n + \cdots + a_1 D_1$: Example (cont'd)

Example 25.36

$$y'' - 2y' = 5, \quad (D^2 - 2D)y = 5.$$

3. Method #1

$$\begin{aligned} y_p &= \frac{1}{D} \left[\frac{1}{D-2}(5) \right] = \frac{1}{D} \left[-\frac{1}{2} \left(1 + \frac{D}{2} + \cdots \right) (5) \right] = \frac{1}{D} \left(-\frac{5}{2} \right) \\ &= -\frac{5}{2}x. \end{aligned}$$

4. Method #2

$$\begin{aligned} y_p &= \frac{1}{D-2} \left[\frac{1}{D}(5) \right] = -\frac{1}{2} \left(1 + \frac{D}{2} + \frac{D^2}{4} + \cdots \right) (5x) \\ &= -\frac{5}{2}x - \frac{5}{4}. \end{aligned}$$

3. $Q(x) = be^{ax}$ and $P(a) \neq 0$

The particular solution is

$$y_p = \frac{1}{P(D)} be^{ax} = \frac{be^{ax}}{P(a)}, \quad P(a) \neq 0.$$

Can be proved by setting the trial function $y_p = Ae^{ax}$ and finding A .
(Try it.)

4. $Q(x) = b \sin ax$ or $b \cos ax$

1. Set $Q(x) = be^{jax}$.
2. Solve it by matching real/imaginary part.

5. $Q(x) = u(x)e^{ax}$, $u(x)$ is a polynomial in x

The particular solution is

$$y_p = \frac{1}{P(D)}u(x)e^{ax} = e^{ax} \frac{1}{P(D+a)}u(x).$$

Can be proved by “exponential shift theorem”:

$$P(D)(u(x)e^{ax}) = e^{ax}P(D+a)(u(x))$$

6. $Q(x) = be^{ax}$ and $P(a) = 0$

1. $P(D) = (D - a)^r F(D)$, $F(a) \neq 0$.
2. The particular solution is

$$\begin{aligned}y_p &= \frac{1}{(D - a)^r F(D)}(be^{ax}) \\ &= \frac{1}{(D - a)^r} \left[\frac{1}{F(D)}(be^{ax}) \right].\end{aligned}$$

3. Applying “type 3” method, we get

$$y_p = \frac{1}{(D - a)^r} \left[\frac{b}{F(a)} e^{ax} \right].$$

4. By the “exponential shift theorem”,

$$y_p = e^{ax} \frac{1}{D^r} \left[\frac{b}{F(a)} \right] = \frac{e^{ax} b x^r}{r! F(a)}, \quad F(a) \neq 0.$$

$$7. Q(x) = Q_1(x) + \cdots + Q_n(x)$$

By “principle of superposition”,

$$y_p = \frac{1}{P(D)}Q(x) \equiv \frac{1}{P(D)}Q_1(x) + \cdots + \frac{a}{P(D)}Q_n(x).$$

Lesson 26: Solution of a Linear Differential Equation by Means of the Par

Partial Fraction Expansion Theorem

Example:

For polynomials $P(x)$ and

$$Q(x) = (x + a)(x^3 + b)(x^2 + c)^2(x + d)^3,$$

where the degree of $P(x)$ is less than $Q(x)$,

$$\frac{P(x)}{Q(x)} = \frac{A}{x + a} + \frac{Bx^2 + Cx + D}{x^3 + b} + \frac{Ex + F}{x^2 + c} + \frac{Gx + H}{(x^2 + c)^2} + \frac{I}{x + d} + \frac{J}{(x + d)^2} + \frac{K}{(x + d)^3}.$$

- ▶ Degree of each numerator?
- ▶ Types of denominators?

Partial Fraction Expansion Theorem (cont'd)

For

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_k) \cdots (x - r_n)$$

where r_1, r_2, \dots, r_n are *distinct*,

$$\frac{1}{f(x)} = \frac{1}{f'(r_1)(x - r_1)} + \cdots + \frac{1}{f'(r_n)(x - r_n)}.$$

Solving a L.D.E. by Means of the Partial Fraction Expansion of Inverse Operators

1. The D.E.:

$$P(D)y = Q(x).$$

2. Partial fraction expand:

$$\frac{1}{P(D)} = \frac{1}{P_1(D)} + \cdots + \frac{1}{P_k(D)}.$$

3. The particular solution is

$$y_p = \frac{Q(x)}{P_1(D)} + \cdots + \frac{Q(x)}{P_k(D)}.$$

Lesson 27: The Laplace Transform. Gamma Function.

Improper Integral

- ▶ $\int_0^{\infty} f(x)dx.$
- ▶ Existence and convergence:

$$\int_0^{\infty} f(x)dx = \lim_{\substack{h \rightarrow \infty \\ \epsilon \rightarrow 0}} \int_{\epsilon}^h f(x)dx = L.$$

Theorem

If the improper integral

$$\int_0^{\infty} e^{-sx}f(x)dx, \quad 0 \leq x < \infty,$$

converges for a value of $s = s_0$, then it converges for every $s > s_0$.

Laplace Transform

$$\mathcal{L}[f(x)] = F(s) = \int_0^{\infty} e^{-sx} f(x) dx.$$

Assuming that $f(x)$ is a function for which the improper integral on the right converges.

Properties:

- ▶ Linear: $\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2]$.
- ▶ Invertible: $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ if and only if $f_1 = f_2$.
→ Inverse Laplace transform: If $\mathcal{L}[f(x)] = F(s)$, then $\mathcal{L}^{-1}\{F(s)\} = f(x)$.

Laplace Transform Method

1. For a L.D.E.

$$a_n y^{(n)}(x) + \cdots + a_1 y'(x) + a_0 y = f(x),$$

we get

$$\begin{aligned} \mathcal{L} [a_n y^{(n)} + \cdots + a_1 y' + a_0 y] \\ = a_n \mathcal{L} [y^{(n)}] + \cdots + a_1 \mathcal{L} [y'] + a_0 \mathcal{L} [y] = \mathcal{L} [f(x)], \quad s > s_0. \end{aligned}$$

2. Assuming (27.28)

$$\lim_{x \rightarrow \infty} e^{-sx} y^{(k)}(x) = 0, \quad k = 0, 1, 2, \dots, n-1, \quad s > s_0,$$

we get

$$\mathcal{L} [y'] = -y(0) + s \mathcal{L} [y].$$

In general, (27.33)

$$\begin{aligned} \mathcal{L} [y^{(n)}] = s^n \mathcal{L} [y] \\ - \left(y^{(n-1)}(0) + s y^{(n-2)}(0) + \cdots + s^{n-2} y'(0) + s^{n-1} y(0) \right). \end{aligned}$$

$\rightarrow s^n \mathcal{L} [y] + \text{Polynomial of } s!$

Laplace Transform Method (cont'd)

3. We can rearrange as (27.4 and 27.41)

$$\mathcal{L}[y] = F(s), \quad s > s_0$$

then

$$y = \mathcal{L}^{-1}\{F(s)\}.$$

Note

- ▶ $F(s)$ can be evaluated if initial conditions $y(0), y'(0), \dots, y^{(n-1)}(0)$ are given.
- ▶ The Laplace transform method has changed the original differential equation involving derivatives, to an algebraic equation involving a function of s .

Laplace Transform of Simple Functions

$$\mathcal{L}[k] = \frac{k}{s} \quad s > 0.$$

$$\mathcal{L}[x^n] = \frac{n!}{s^{n+1}} \quad s > 0, n = 1, 2, \dots$$

$$\mathcal{L}[e^{ax}] = \frac{1}{s-a}, \quad s > a.$$

$$\mathcal{L}[\sin ax] = \frac{a}{s^2 + a^2}.$$

...and more. (See the table on p.306)

Theorem

If $F(s) = \mathcal{L}[f(x)]$, $s > s_0$,

$$F^{(n)}(s) = (-1)^n \mathcal{L}[x^n f(x)] = (-1)^n \int_0^{\infty} e^{-sx} x^n f(x) dx, \quad s > s_0.$$

→ $\mathcal{L}[x^n f(x)]$ can be obtained from $\mathcal{L}[f(x)]$.

Faltung (Folding) Theorem

Theorem

If

$$F(s) = \mathcal{L}[f(x)] \quad \text{and} \quad G(s) = \mathcal{L}[g(x)],$$

then

$$\begin{aligned} \mathcal{L}\left[\int_0^x f(x-t)g(t)dt\right] &= \mathcal{L}\left[\int_0^x f(t)g(x-t)dt\right] \\ &= \mathcal{L}[f(x)] \cdot \mathcal{L}[g(x)] = F(s) \cdot G(s). \end{aligned}$$

Laplace Transform of $x^{1/2}$ and $x^{n-1/2}$?

We need Gamma function!

The Gamma Function

- ▶ Extension of factorial function to real and complex numbers.

- ▶ $\Gamma(k) = \int_0^{\infty} x^{k-1} e^{-x} dx, \quad k > 0.$

- ▶ $\Gamma(1) = 1$ and $\Gamma(k+1) = k\Gamma(k)$
→ $\Gamma(n+1) = n!$ n integer.

- ▶ $\Gamma(k) = \frac{\Gamma(k+n)}{k(k+1)\cdots(k+n-1)}, \quad k \neq 0, -1, \dots, -(n-1).$

→ Extended to negative values of k , provided that $k \neq 0, -1, \dots, -(n-1).$

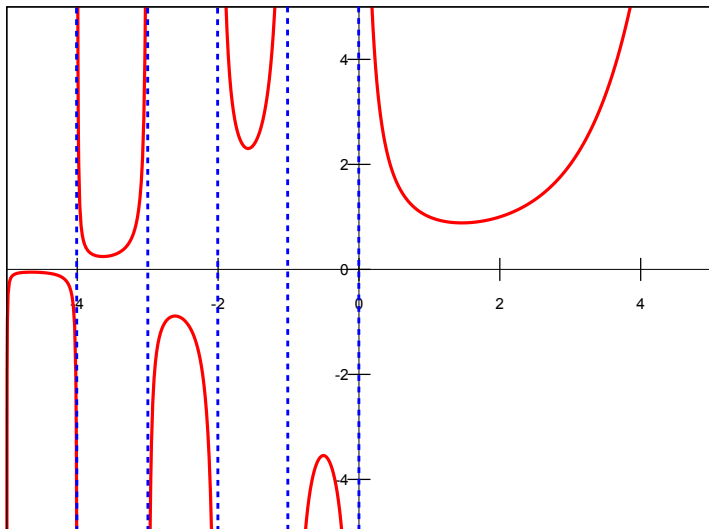
Example: $\Gamma(-1/2) = -2\Gamma(1/2).$

- ▶ Euler's reflection formula

$$\Gamma(1-k)\Gamma(k) = \frac{\pi}{\sin(\pi k)}$$

The Gamma Function (cont'd)

Gamma function



Laplace Transform and the Gamma Function

$$\mathcal{L}[x^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

→ The Laplace transform $\mathcal{L}[x^n] = \frac{n!}{s^{n+1}}$ $s > 0$, can be extended to any n , except negative integers.