Mathematical Models for Engineering Problems and Differential Equations

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Chapter 4: Linear Differential Equations of Order Greater Than One

Linear DE of order n

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x)$$

- $f_0(x), f_1(x), \cdots, f_n(x)$ and $Q(x)$ are continuous functions of x defined on a common interval I and
- $f_n(x) \not\equiv 0$ in I. (The order is not n otherwise.)
- *Homogeneous* if $Q(x) \equiv 0$ and *nonhomegeneous* if $Q(x) \not\equiv 0$.

Lesson 18: Complex Numbers and Complex Functions.

Complex numbers

Definition

$$z = x + yj$$

- j is defined by the relation $j^2 = -1$
- x: real part of z
- y: imaginary part of z

Complex numbers (cont'd)

Conjugate of z

$$\overline{z} = x - yj$$

Absolute value of z

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}$$

Polar form

$$z = r(\cos\theta + j\sin\theta)$$

- $r = |z|$
- $\theta = \text{Arg}z$ (argument of z) is defined to be the *smallest* positive angle satisfying

$$\cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}$$

Algebra of complex numbers

- $z_1 + z_2 =?$
- $z_1 - z_2 =?$
- $z_1 z_2 =?$
- $z_1 / z_2 =?$

Exponential, trigonometric, and hyperbolic functions of complex numbers

By the *Taylor* (or *Maclaurin*) series expansions of e^x, $\sin x$ and $\cos x$,

$$
\begin{aligned}
e^z &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \\
\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots
\end{aligned}
$$

Exponential, trigonometric, and hyperbolic functions of complex numbers (cont'd)

$$
\begin{aligned}
e^{0} &= 1 \\
e^{z_1} e^{z_2} &= e^{z_1 + z_2} \\
e^{j\theta} &= \cos\theta + j\sin\theta \\
e^{-j\theta} &= \cos\theta - j\sin\theta \\
\sin\theta &= \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \\
\cos\theta &= \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \\
e^{z} &\neq 0 \quad \text{for any value of } z
\end{aligned}
$$

Exponential, trigonometric, and hyperbolic functions of complex numbers (cont'd)

Hyperbolic functions

$$\sinh z = \frac{e^z - e^{-z}}{2}$$
$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

Lesson 19: Linear Independence of Functions. The Linear DE of Order n.

Linear independence of functions

Definition

A set of functions $f_1(x), f_2(x), \cdots, f_n(x)$, each defined on a common interval I, is called *linearly dependent* on I, if there exists a set of constants $c_1, c_2, \cdots, c_n$, not *all zeros*, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for *every* x on I.
$\rightarrow$ Any function in the set can be expressed as a linear combination of the rest.

Linear independence and linear DE

- A homogeneous linear DE has as many linearly independent solutions as the order of its equation. (Theorem 19.3)
- For a linear DE of order n, we need to
  - find out n solutions and
  - show these n solutions are linearly independent.

Existance and uniqueness theorem

Existance and uniqueness theorem

If $f_0(x), f_1(x), \cdots, f_n(x)$ and $Q(x)$ are each continuous functions of x on a common interval I, and $f_n(x) \neq 0$ when x is on I, then the linear differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x)$$

has one and only one solution

$$y = y(x),$$

satisfying the set of initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \cdots \quad , y^{(n-1)}(x_0) = y_{n-1},$$

where x_0 is in I, and $y_0, y_1, \cdots, y_{n-1}$ are constants.

- Proof in Theorem 65.2

Three important properties

1. The homogeneous linear differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y = 0 \qquad (1)$$

   has n linearly independent solutions $y_1(x), y_2(x), \cdots, y_n(x)$.

2. The linear combination of these n solutions

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

   where $c_1, c_2, \cdots, c_n$ is a set of n arbitrary constants, is also a solution of (1). (*complementary function* of (1))

3. The function

$$y(x) = y_c(x) + y_p(x),$$

   where $y_p(x)$ is a particular solution of the nonhomogeneous linear differential equation corresponding to (1), namely

$$f_n(x)y^{(n)} + f_{n-1}y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x), \qquad (2)$$

   is an n-parameter family of solutions of (2).

Overview of methods

1. Characteristic equation (Lesson 20)
   - *Homogeneous* linear DE of order n with constant coefficients
   - Three cases
2. Method of undetermined coefficients (Lesson 21)
   - *Nonhomogeneous* linear DE of order n with *constant* coefficients
   - $Q(x)$ consists of a sum of terms each of which has a *finite* number of linearly independent derivatives, e.g., a, x^k, e^{ax}, $\sin ax$, $\cos ax$, etc. (a constant, k positive integer)
   - Three cases
3. Variation of parameters (Lesson 22)
   - $Q(x)$ has an *infinite* number of linearly independent derivatives.
4. Reduction of order method (Lesson 23)
   - Nonhomogeneous linear DE of order n with *nonconstant* coefficients

Lesson 20: Solution of the Homogeneous Linear DE of Order n with Cons

Method #1: Characteristic equation

- For *homogeneous* linear DE of order n with *constant* coefficients ($a_n \neq 0$):

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

- If we assume that a possible solution has the form

$$y = e^{mx},$$

we get the *characteristic equation*

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

- Three cases
  1. All roots are distinct and real. (Lesson 20B)
  2. All roots are real but some repeat. (Lesson 20C)
  3. All roots are imaginary. (Lesson 20D)

Case 1: All roots are real and distinct

The general solution is

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

Why?

Case 2: All roots are real but some repeat.

- Find $u(x)$ such that $y_c = u(x)e^{ax}$ is a solution of the DE.
  $\rightarrow$ Find $u(x)$ for 2nd order case!
- If the characteristic equation has a root $m = a$ of multiple k, then $e^{ax}, xe^{ax}, \cdots, x^{k-1}e^{ax}$ are k linearly independent solutions.

Case 3: All roots are imaginary.

The general solution of a linear DE of order 2, whose characteristic equation has the conjugate roots $\alpha \pm \beta j$, can be written in any of the following forms:

- $y_c = c_1 e^{(\alpha+\beta j)x} + c_2 e^{(\alpha-\beta j)x}$
- $y_c = c^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
- $y_c = ce^{ax} \sin(\beta x + \delta)$
- $y_c = ce^{ax} \cos(\beta x - \delta)$

Lesson 21: Solution of the Nonhomogeneous Linear DE of Order n with C

Method #2: Method of undetermined coefficients

- For *nonhomogeneous* linear DE with constant coefficients.
- Can be used only if $Q(x)$ consists of a sum of terms each of which has a finite number of linearly independent derivatives, e.g., a, x^k, e^{ax}, $\sin ax$, $\cos ax$, etc. (a constant, k positive integer)
- Three cases:
  1. When no term in $Q(x)$ is the same as a term in y_c
  2. When $Q(x)$ contains a term which, ignoring constant coefficients, is x^k times a term $u(x)$ of y_c (k nonnegative integer).
  3. When both of the following conditions are fulfilled:
     - The characteristic equation of the given DE has an r multiple root.
     - $Q(x)$ contains a term which, ignoring constant coefficients, is x^k times a term $u(x)$ in y_c, where $u(x)$ was obtained from the r multiple root.

Case 1

When no term in $Q(x)$ is the same as a term in y_c
$\rightarrow$ A particular solution y_p will be a linear combination of the terms in $Q(x)$ and *all* its linearly independent derivatives.

Case 1: Example 21.2

$$y'' + 4y' + 4y = 4x^2 + 6e^x$$

1. Since
   - the coefficients are constants and
   - the linearly independent derivatives of $Q(x)$ are x^2, x, 1 and e^x, i.e., $Q(x)$ consists of terms with finite number of linearly independent derivatives,

   we can apply the "method of undetermined coefficients."
2. The complementary function obtained by solving the homogeneous D.E. is $y_c = (c_1 + c_2x)e^{-2x}$
   $\rightarrow y_c$ is composed of e^{-2x} and xe^{-2x}.
3. Terms in $Q(x)$ are x^2 and e^x, therefore no term of $Q(x)$ is the same as a term of y_c
   $\rightarrow$ This is case 1.

Case 1: Example 21.2 (cont'd)

4. A particular solution is a linear combination of the terms in $Q(x)$ and all its linearly independent derivatives, i.e., $y_p = Ax^2 + Bx + C + De^x$.
5. Substitute y in the D.E. with y_p to determine A, B, C and D.
6. The general solution is $y = y_c + y_p$.

Case 2

When $Q(x)$ contains a term which, ignoring constant coefficients, is x^k times a term $u(x)$ of y_c (k nonnegative integer).
$\rightarrow$ A particular solution y_p will be a linear combination of $x^{k+1}u(x)$ and all its linearly independent derivatrives (ignoring constant coefficients). If in addition $Q(x)$ contains terms which belong to Case 1, then the proper terms called for by this case must be included in y_p.

Case 2: Example 21.32

$$y'' + y = \sin^3 x$$

1. First, $Q(x) = \sin^3 x = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x$.
2. Since
   - the coefficients are constants and
   - the linearly independent derivatives of $Q(x)$ are $\sin 3x$, $\cos 3x$, $\sin x$, and $\cos x$, i.e., $Q(x)$ consists of terms with finite number of linearly independent derivatives,

   we can apply the "method of undetermined coefficients."
3. The complementary function of the homogeneous D.E. is $y_c = c_1 \sin x + c_2 \cos x$.

Case 2: Example 21.32 (cont'd)

4. For each term of $Q(x)$,
   - the term $\sin 3x$ of $Q(x)$ is not in $y_c \rightarrow$ Case 1. A particular solution is a linear combination of $\sin 3x$ and all its linearly independent derivatives, $\sin 3x$ and $\cos 3x$.
   - the term $\sin x$ of $Q(x)$ is x^0 ($k = 0$) times a term, $u(x) = \sin x$, of y_c. $\rightarrow$ Case 2. A particular solution is a linear combination of $x^{k+1}u(x) = x \sin x$ and all its linearly independent derivatives, $x \sin x$, $x \cos x$, $\sin x$ and $\cos x$.

   $\rightarrow y_p = A \sin 3x + B \cos 3x + Cx \sin x + Dx \cos x$. ($\sin x$ and $\cos x$ are excluded since they are already in y_c.)
5. Substitute y in the D.E. with y_p to determine A, B, C and D.
6. The general solution is $y = y_c + y_p$.

Case 3

When both of the following conditions are fulfilled:

- The characteristic equation of the given DE has an r multiple root.
- $Q(x)$ contains a term which, ignoring constant coefficients, is x^k times a term $u(x)$ in y_c, where $u(x)$ was obtained from the r multiple root.

$\rightarrow$ A particular solution y_p will be a linear combination of $x^{k+r}u(x)$ and all its linearly independent derivatives. If in addition $Q(x)$ contains terms which belong to Case 1 and 2, then the proper terms called for by these cases must also be added to y_p.

Case 3: Example 21.4

$$y'' + 4y' + 4y = 3xe^{-2x}$$

1. Since
   - the coefficients are constants and
   - the linearly independent derivatives of $Q(x)$ are xe^{-2x} and e^{-2x}, i.e., $Q(x)$ consists of terms with finite number of linearly independent derivatives,

   we can apply the "method of undetermined coefficients."
2. The complementary function obtained by the homogeneous D.E. is $y_c = (c_1 + c_2 x)e^{-2x}$.
3. Note that
   - the characteristic equation of the homogeneous D.E. has $r = 2$ multiple root, -2, and
   - $Q(x)$ contains a term, xe^{-2x}, which is $x^{k_1} = x^0$ (or $x^{k_2} = x^1$) times a term $u_1(x) = xe^{-2x}$ (or $u_2(x) = e^{-2x}$) of y_c.

   $\rightarrow$ Case 3.

Case 3: Example 21.4 (cont'd)

4. A particular solution if a linear combination of $x^{k_1+r}u_1(x) = x^2 xe^{-2x}$ (or $x^{k_2+r}u_2(x) = x^3e^{-2x}$) and all its linearly independent derivatives, x^3e^{-2x}, x^2e^{-2x}, xe^{-2x} and e^{-2x}.
$\rightarrow y_p = Ax^3e^{-2x} + Bx^2e^{-2x}$ (xe^{-2x} and e^{-2x} are excluded since they are already in y_c.)
5. Substitute y in the D.E. with y_p to determine A and B.
6. The general solution is $y = y_c + y_p$.

Lesson 22: Solution of the Nonhomogeneous Linear DE by the Method of

Method #3: Method of variation of parameters

- $Q(x)$ contains terms whose linearly independent derivatives are *infinite* in number.
- A particular solution

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$$

of the nonhomogeneous liear DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = Q(x), \quad a_n \neq 0$$

can be obtained from n linearly independent solutions of its related homogeneous equation, $y_1, y_2, \cdots, y_n$, and $u_1', u_2', \cdots, u_n'$ are the functions obtained by solving simultaneously the following set of equations:

$$\begin{aligned} u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n &= 0 \\ u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' &= 0 \\ &\cdots\cdots\cdots\cdots \\ u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} &= \frac{Q(x)}{a_n}. \end{aligned}$$

2nd Order Case

$$a_2 y'' + a_1 y' + a_0 y = Q(x), \quad a_2 \neq 0$$

1. Assume that we already have found two linearly independent solutions y_1 and y_2 of the homogeneous D.E.
2. Set a particular solution as $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.
3. Substitute y_p, y_p' and y_p'' in the D.E. then we get

$$a_2(u_1'y_1' + u_2'y_2') + a_2(u_1'y_1 + u_2'y_2)' + a_1(u_1'y_1 + u_2'y_2) = Q(x).$$

4. The above equation holds if

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + +u_2'y_2' &= \frac{Q(x)}{a_2}. \end{aligned}$$

5. After finding u_1' and u_2' from the above equations, we can get u_1 and u_2 via integration.

Example 22.4

$$y'' - 3y' + 2y = \sin e^{-x}$$

with $y_c = c_1 e^x + c_2 e^{2x}$.

1. There are infinitely many linearly independent derivatives of $Q(x)$: $\sin e^{-x}$, $e^{-x} \cos e^{-x}$, $e^{-x} \sin e^{-x}$, ... Therefore we can apply the “method of variation of parameters.”
2. Set $y_p = u_1 e^x + u_2 e^{2x}$.
3. By solving

$$u_1' e^x + u_2' e^{2x} = 0$$
$$u_1' e^x + u_2'(2e^{2x}) = \sin e^{-x}$$

we get $u_1' = -e^{-x} \sin e^{-x}$ and $u_2' = e^{-2x} \sin e^{-x}$.

Example 22.4 (cont'd)

4. Therefore

$$u_1 = \int u_1'(x)dx = -\cos e^{-x}$$
$$u_2 = \int u_2'(x)dx = -\sin e^{-x} + e^{-x}\cos e^{-x}.$$

and $y_p = u_1 y_1 + u_2 y_2$.

Lesson 23: Solution of the Linear DE with Nonconstant Coefficients. Red

Method #4: Reduction of order method

- For the homogeneous linear DE

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = 0$$

  with nonconstant coefficients, we can find one independent solution if the other $n-1$ independent solutions are known.
- For second order DE, an independent solution y_2 has the form

$$y_2(x) = y_1(x)\int u(x)dx$$

  where $y_1(x)$ is the known solution and

$$u(x) = \frac{e^{-\int \frac{f_1(x)}{f_2(x)}dx}}{y_1^2}.$$

  A particular solution can be found by substituting $y_2(x)$ in the nonhomogeneous linear DE.

2nd Order Case

$$f_2(x)y'' + f_1(x)y' + f_0(x) = Q(x)$$

1. Assume that one solution of the homogeneous D.E. is known, say, $y_1(x)$.
2. Assume that the other solution is of the form

$$y_2(x) = y_1(x) \int u(x)dx.$$

3. Substitute y_2, y_2' and y_2'' in the D.E. then we get

$$f_2(x)y_1u' + [2f_2(x)y_1' + f_1(x)y_1]u = 0.$$

4. Multiplying both sides by $dx/[uf_2(x)y)1]$ we get

$$\frac{du}{u} + \frac{2dy_1}{y_1} = \frac{-f_1(x)}{f_2(x)}dx.$$

2nd Order Case (cont'd)

5. By solving the above equation we obtain

$$u(x) = \frac{e^{-\int \frac{f_1(x)}{f_2(x)} dx}}{y_1^2}$$

and $y_2(x) = y_1(x) \int u(x) dx$.