# Mathematical Models for Engineering Problems and Differential Equations

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# Introductory remarks

1. Only very special types of $1$st order DE possess solutions which can be expressed in terms of the elementary functions.
2. The "look" of a DE doesn't tell you how easy (or difficult) it is to solve it.
3. Implicit solutions usually are not practical.
4. The solution you find may be *extraneous*. Therefore you should always verify that it does in fact satisfy the given DE.
5. The examples in the textbook are only for illustration and not from real applications.

# Table of contents

Chapter 2: Special Types of Differential Equations of the First Order
Lesson 6: Meaning of the Differential of a Function. Separable Differe
Lesson 7: First Order Differential Equation with Homogeneous Coeffic
Lesson 8: Differential Equations with Linear Coefficients
Lesson 9: Exact Differential Equations
Lesson 10: Recognizing Exact Differential Equations. Integrating Fact
Lesson 11: The Linear Differential Equation of the First Order. Berno
Lesson 12: Miscellaneous Methods of Solving a First Order Differentia

Lesson 6: Meaning of the Differential of a Function. Separable Differenti

# Increment ($\Delta y$) and differential ($dy$)

Let $y = f(x)$ define $y$ as a function of $x$.

- Increament: "How much $y$ increases from $x$ to $x + \Delta x$?"
  $\Delta y = (\Delta f)(x, \Delta x) = f(x + \Delta x) - f(x)$
- Differential: Approximates $\Delta y$ using $f'(x)$.
  $dy = (df)(x, \Delta x) = f'(x)\Delta x$

$\rightarrow (dy)(x, \Delta x) = f'(x)(d\hat{x})(x, \Delta x)$
$\rightarrow dy = f'(x)dx$

# Type #1: $f(x)dx + g(y)dy = 0$

A 1-parameter family of solutions of the DE with **separable variables**,

$$f(x)dx + g(y)dy = 0,$$

is

$$\int f(x)dx + \int g(y)dy = C.$$

Lesson 7: First Order Differential Equation with Homogeneous Coefficien

# Homogeneous function

### Definition

The function $f(x,y)$ is $n$**-th order homogeneous** if it can be written as

$$f(x,y) = x^n g(u), u = y/x$$

or

$$f(x,y) = y^n h(u), u = x/y$$

Alternatively, $f(x,y)$ is homogeneous of order $n$ if

$$f(tx,ty) = t^n f(x,y).$$

## Type #2: Homogeneous $P(x,y)dx + Q(x,y)dy = 0$

The 1st order DE with $n$-th order homogeneous coefficients,

$$P(x,y)dx + Q(x,y)dy = 0,$$

can be converted to the equation with separable variables (Type #1)

$$\frac{dx}{x} + \frac{g_2(u)}{g_1(u) + ug_2(u)}du = 0, x \neq 0, g_1(u) + ug_2(u) \neq 0,$$

where $P(x,y) = x^n g_1(u)$, $Q(x,y) = x^n g_2(u)$, and $u = y/x$. Alternatively, we can convert by substituting $u = x/y$.

# Lesson 8: Differential Equations with Linear Coefficients

# Type #3: $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$

1. Two lines are not parallel.
   The DE with linear coefficients

   $$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

   can be converted to the equation with homogeneous coefficients

   $$(a_1\bar{x} + b_1\bar{y})d\bar{x} + (a_2\bar{x} + b_2\bar{y})d\bar{y} = 0$$

   where $\bar{x} = x - h$, $\bar{y} = y - k$, and $(h, k)$ is the unique intersection point of the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.
2. Two lines are parallel or coincide.

Lesson 9: Exact Differential Equations

# Type #4: Exact $P(x,y)dx + Q(x,y)dy = 0$

A DE
$$P(x,y)dx + Q(x,y)dy = 0$$
is **exact** if there exists a function $f(x,y)$ such that
$$P(x,y) = \frac{\partial f(x,y)}{\partial x}$$
and
$$Q(x,y) = \frac{\partial f(x,y)}{\partial y}.$$
A 1-parameter family of solutions of this equation is
$$f(x,y) = c.$$

# Type #4: Exact $P(x,y)dx + Q(x,y)dy = 0$ (cont'd)

- Necessary and sufficient condition for exactness:

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}.$$

- The 1-parameter solution:

$$f(x,y) = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy = c$$

or

$$f(x,y) = \int_{y_0}^{y} Q(x,y)dy + \int_{x_0}^{x} P(x,y_0)dx = c.$$

# Lesson 10: Recognizing Exact Differential Equations. Integrating Factors.

# Integrable combinations

- List of known exact DEs.
- Some exact DEs can be easily solved by rearranging the terms to extract integrable combinations.

# Type #5: Integrating factors

## Definition

A multiplying factor which will convert an inexact DE into an exact one is called an **integrating factor**.

## Remark

Theoretically an integrating factor exists for every DE of the form $P(x,y)dx + Q(x,y)dy = 0$, but no general rule is known to discover it.

1. $h = h(x)$
2. $h = h(y)$
3. $h = h(u), u = xy$
4. $h = h(u), u = x/y$
5. $h = h(u), u = y/x$

# 1. Integrating factor $h(x)$

$$h(x) = e^{\int F(x)dx}$$

where

$$F(x) = \frac{\frac{\partial}{\partial y}P(x,y) - \frac{\partial}{\partial x}Q(x,y)}{Q(x,y)}.$$

# 2. Integrating factor $h(y)$

$$h(y) = e^{\int G(y)dy}$$

where

$$G(y) = \frac{\frac{\partial}{\partial x}Q(x,y) - \frac{\partial}{\partial y}P(x,y)}{P(x,y)}.$$

# 3. Integrating factor $h(u)$, $u = xy$

$$h(u) = e^{\int F(u)du}$$

where

$$F(u) = \frac{\frac{\partial}{\partial y}P(x,y) - \frac{\partial}{\partial x}Q(x,y)}{yQ(x,y) - xP(x,y)}.$$

# 4. Integrating factor $h(u)$, $u = x/y$

$$h(u) = e^{\int G(u)du}$$

where

$$G(u) = \frac{y^2 \left[\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}\right]}{xP(x,y) + yQ(x,y)}.$$

# 5. Integrating factor $h(u)$, $u = y/x$

$$h(u) = e^{\int K(u)du}$$

where

$$K(u) = \frac{x^2\left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right]}{xP(x,y) + yQ(x,y)}.$$

# Special form

The integrating factor of the DE of the form

$$y(Ax^p y^q + Bx^r y^s)dx + x(Cx^p y^q + Dx^r y^s)dy = 0$$

is

$$x^a y^b.$$

Lesson 11: The Linear Differential Equation of the First Order. Bernoulli

## Type #6: $\frac{dy}{dx} + P(x)y = Q(x)$

The integrating factor of the **linear differential equation of the** $1$**st order**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is

$$e^{\int P(x)dx}$$

and the solution is

$$y = e^{-\int P(x)dx} \int e^{\int P(x)dx} Q(x)dx + ce^{-\int P(x)dx}.$$

### Note

For the $1$-st order linear DE, the $1$-parameter family of solutions is a true general solution.

# Type #7: Bernoulli equation

The Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

can be converted to Type #6 by multiplying $(1-n)y^{-n}$.

Lesson 12: Miscellaneous Methods of Solving a First Order Differential E