Mathematical Models for Engineering Problems and Differential Equations

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Chapter 1: Basic Concepts

Lesson 1: How Differential Equations Originate

Where do the DEs arise?

"world of interrelated changing entities"

- the position of the earth changes with time
- the velocity of a falling body changes with distance
- the bending of a beam changes with the weight of the load placed on it
- the area of a circle changes with the size of the radius
- the path of a projectile changes with the velocity and angle at which it is fired

Differential equations

- Terms
  - Variables: changing entities
  - Derivative: the rate of change of one variable with respect to another
  - Differential equations: equations which express a relationship among these variables and their derivatives
- Differential equations originate whenever a universal law is expressed by means of variables and their derivatives.
- We are interested in "the problems of determining a relationship among the variables from the information given to us about themselves and their derivatives."

Steps to find a (universal) law

1. Build a model (usually in differential equations) assuming some relations.
2. Find the solution.
3. Make some predictions from the model.
4. Validate the model by experiments.
5. Accept the model if the result is correct.

note

This depends on the accuracy of the experiments. (e.g. laws of Newton vs. General Relativity and Relativistic Quantum Mechanics)
http://en.wikipedia.org/wiki/Newton%27s_laws_of_motion

Example: (Carbon-14 test) To determine from the charcoal remains how long the tree died.

- What we know
  - All living organisms contain two isotopes of carbon: C^{12} (stable) & C^{14} (radioactive).
  - The ratio of C^{12} & C^{14} is constant in any macroscopic piece of *living* organism.
  - Since the living organism is dead, the amount of C^{14} decreases and is not replaced.
- Variables
  - t: time (years past since the tree is dead)
  - x: the amount of C^{14} present in the dead tree at t

Example (cont'd)

- Procedure
  1. Builds a model (differential equation):

$$\frac{dx}{dt} = -kx, (k > 0, \text{ constant})$$

  2. Finds the solution:

$$x(t) = Ae^{-kt}$$

  3. Finds the answer using additional clues.
     (i) Approximately 99.876% of C^{14} present at death will remain in dead wood after 10 years.

$$x(10) = Ae^{-10k} = 0.99876A \rightarrow k = 0.000124$$

     (ii) 85.5% of the amount of C^{14} present at death had decomposed.

$$0.145A = Ae^{-0.000124t} \rightarrow t = 15573(\text{years})$$

Lesson 2: The Meaning of the Terms *Set* and *Functions*. Implicit Functio

Implicit function

Definition

The function $y = g(x)$ is **implicit** if the value of y is obtained from x by *solving* an equation of the form:

$$f(x, y) = 0.$$

(http://en.wikipedia.org/wiki/Implicit_function)

- Implicit function theorem
- Example

  $$f(x, y) = x^2 + y^2 - 25 = 0$$

  defines y as an implicit function of x on the interval $I : -5 \leq x \leq 5$.

Lesson 3: The Differential Equation

Ordinary Differential Equation (ODE)

Definition

Let $f(x)$ define a function of x on an interval $I : a < x < b$. By an **ordinary differential equation** we mean an equation involving x, the function $f(x)$ and one or more of its derivatives.

Order of a DE

The order of the highest derivative involved in the equation.

Examples

$$\frac{dy}{dx} + y = 0$$

$$y' = e^x$$

$$\frac{d^2y}{dx^2} = \frac{1}{1 - x^2}$$

$$f'(x) = f''(x)$$

$$xy' = 2y$$

$$y'' + (3y')^3 + 2x = 7$$

$$(y''')^2 + (y'')^4 + y' = x$$

$$xy^{(4)} + 2y'' + (xy')^5 = x^3$$

Explicit solution

Definition

Let $y = f(x)$ define y as a function of x on an interval $I : a < x < b$. $f(x)$ is an **explicit solution** of the ODE

$$F(x, y, y', \cdots, y^{(n)}) = 0$$

if

$$F[x, f(x), f'(x), \cdots, f^{(n)}(x)] = 0.$$

Example:
$y = x^2$, $-\infty < x < \infty$, is a solution of the ODE

$$(y'')^3 + (y')^2 - y - 3x^2 - 8 = 0.$$

Implicit solution

Definition

A "relation" $f(x, y) = 0$ will be called an **implicit solution** of the ODE

$$F(x, y, y', ..., y^{(n)}) = 0$$

on an interval $I : a < x < b$, if

1. it defines y as an implicit function of x on I, i.e., if there exists a function $g(x)$ defined on I such that $f[x, g(x)] = 0$ for *every* x in I, and
2. $g(x)$ is an explicit solution.

Example:
$f(x, y) = x^2 + y^2 - 25 = 0$ is an implicit solution of the ODE

$$F(x, y, y') = yy' + x = 0.$$

for every x in I.

Lesson 4: The General Solution of a Differential Equation

Solutions of a DE

- Examples
  - The solution of "$y' = e^x$" is $y = e^x + c$.
  - The solution of "$y'' = e^x$" is $y = e^x + c_1x + c_2$.
  - The solution of "$y''' = e^x$" is $y = e^x + c_1x^2 + c_2x + c_3$.
- Are the followings true?
  1. If an ODE has a solution, it has infinitely many of them.
     No. Counter example: The DE $(y')^2 + y^2 = 0$ has only the one solution, $y = 0$.
  2. The solution of an n-th order ODE contains n arbitrary constants.
     No. Counter example: $(y' - y)(y' - 2y) = 0$ has the solution $(y - c_1e^x)(y - c_2e^{2x}) = 0$,
- But, in most cases they are true!

n-parameter family of solutions

Definition

The functions defined by

$$y = f(x, c_1, c_2, \cdots, c_n)$$

of the $n+1$ variables will be called an n-**parameter family of solutions** of the n-th order ODE

$$F(x, y, y', \cdots, y^{(n)}) = 0$$

if for each choice of a set of values c_1, c_2, $\cdots$, c_n,

$$F(x, f, f', \cdots, f^{(n)}) = 0.$$

"A ODE of the n-th order has an n-parameter family of solutions."

Particular solution and general solution

Definition: Particular solution

A solution of a DE will be called a **particular solution** if it satisfied the equation and does not contain arbitrary constants.

Definition: General solution

An n-parameter family of solutions of a DE will be called a **general solution** if it contains *every* particular solution.

General solution and singular solution

"n-parameter family of solution" is not always a general solution!

- Example #1
  ODE: $y = xy' + (y')^2$
  1-parameter family of solutions: $cx + c^2$
  but $y = -x^2/4$ is also a solution which cannot be obtained from the above! $\rightarrow$ traditionally called a "singular solution."
- Example #2
  ODE: $y' = -2y^{\frac{3}{2}}$
  1-parameter family of solutions: $y = 1/(x+c)^2$
  singular solution: $y = 0$
  but we can express both by $y = C^2/(Cx+1)^2$ $\rightarrow$ A solution can be both singular and nonsingular, depending on the choice of representation of the 1-parameter family.

Initial conditions

Definition

The n conditions which enable us to determine the values of the arbitrary constants c_1, c_2, $\cdots$, c_n in an n-parameter family of solutions, if given in terms of one value of the independent variable, are called **initial conditions**.

- Normally the number of initial conditions must equal the order of the DE.

Lesson 5: Direction Field

Direction field

Geometric significance of a solution of a 1st order DE

Finding a 1-parameter family of solutions of

$$y' = F(x, y), a < x < b$$

means to find a family of curves (integral curves), every member of which has at each of its points a slope given by the DE.

- A **direction field** can be constructed by drawing **line elements**.
- Example: Fig 5.22 is the direction field of the DE $y' = x + y$.

Isoclines

Easier way to find the direction field

If

$$y' = F(x, y),$$

then each curve for which

$$F(x, y) = k$$

will be an **isocline** of the direction field. Every integral curve will cross the isocline with a slope k.

Example:

$$x^2 + y^2 = c(c >= 0)$$
$$\rightarrow x + yy' = 0 \rightarrow y' = -x/y \rightarrow k = -x/y$$

The integral curves cross the isocline $y = -(1/k)x$ with a slope k $\rightarrow$ They are orthogonal.

Ordinary point

Definition

An **ordinary point** of the 1st order DE

$$y' = F(x, y), a < x < b$$

is a point in the plane which lies on *one and only one* of the integral curves.

Singular point

Definition

A **singular point** of the 1st order DE

$$y' = F(x, y), a < x < b$$

is a point in the plane which meets the following two requirements:

1. It is not an ordinary point
2. If a circle of arbitrarily small radius is drawn about the point, there is at least one ordinary point in its interior. (the singular point is a "limit" of ordinary points)
   $\rightarrow$ to exclude "extraneous points" (e.g. the point $(3, 7)$ is an extraneous point of the DE $y' = \sqrt{1 - x^2}$.)