- Exercise 19.1 (p.210)

  If e^{px} and e^{qx} are linearly dependent, there exist non-zeros constants c_1 and c_2 such that

  $$c_1 e^{px} + c_2 e^{qx} = 0,$$

  therefore we get

  $$e^{(p-q)x} = -\frac{c_2}{c_1}, \quad p \neq q.$$

  Since $p \neq q$, the left-hand side of the equation varies with x except $x = 0$, while the right-hand side is a constant, therefore the assumption leads to a constradiction.

- Exercise 20-24 (p.220)

  Let the solution be of the form e^{mx}, then the characteristic equation is

  $$m^2 - m + 1 = 0,$$

  of which roots are

  $$m_1 = \frac{1+\sqrt{3}j}{2} \quad \text{and} \quad m_2 = \frac{1-\sqrt{3}j}{2}.$$

  Therefore, the general solution is

  $$\begin{aligned} ay &= c_1 e^{\frac{1+\sqrt{3}j}{2}x} + c_2 e^{\frac{1-\sqrt{3}j}{2}x} \\ &= e^{x/2}\left(c_1\left(\cos\frac{\sqrt{3}}{2}x + j\sin\frac{\sqrt{3}}{2}x\right) c_2\left(\cos\frac{\sqrt{3}}{2}x - j\sin\frac{\sqrt{3}}{2}x\right)\right) \\ &= e^{x/2}\left((c_1+c_2)\cos\frac{\sqrt{3}}{2}x + (c_1-c_2)j\sin\frac{\sqrt{3}}{2}x\right) \\ &= e^{x/2}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right), \quad c_3 := c_1 + c_2, c_4 := (c_1 - c_2)j. \end{aligned}$$

- Exercise 21-26 (p.231)

  First we claim that solving the differential equation

  $$y^{(5)} + 2y''' + y' = \sin x + \cos x \tag{1}$$

  is equivalent to solving the differential equation

  $$y^{(5)} + 2y''' + y' = e^{jx}. \tag{2}$$

  Let $\mathrm{Re}\,(z(x))$ and $\mathrm{Im}\,(z(x))$ be the real and imaginary part of a complex function $z(x) = z_r(x) + jz_i(x)$, where $z_r(x)$ and $z_i(x)$ are real functions. Let $Y(x)$ be the general solution of the differential equation (2). Then, since $z'(x) = z_r'(x) + jz_i'(x)$, $\mathrm{Re}\,(Y(x))$ and $\mathrm{Im}\,(Y(x))$ are the general solutions of the differential equations

$$y^{(5)} + 2y''' + y' = \cos x$$

and

$$y^{(5)} + 2y''' + y' = \sin x,$$

respectively. Threfore, the genera solution of (1) is $\mathrm{Re}\,(Y(x)) + \mathrm{Im}\,(Y(x))$.

For the homogeneous differential equation

$$y^{(5)} + 2y''' + y' = 0,$$

the characteristic equation is

$$m^5 + 2m^3 + m = 0,$$

of which roots are $\pm j$, each of which is of multiple 2, and 0. Therefore the complementary solution is

$$y_c(x) = c_0 + c_1 e^j + c_2 x e^j + c_3 e^{-j} + c_4 x e^{-j}.$$

Now consider the differential equation

$$y^{(5)} + 2y''' + y' = 2x + e^{jx}.$$

1. For the term $2x$ on the right-hand side, since this term is x^1 times of the term c_0 of y_c, this is the case 2 on p.224, with $k = 1$. Therefore the particular solution contains $x^{k+1} = x^2$ and all its linearly independent derivatives; x^2, x and 1.
2. For the term e^{jx} on the right-hand side, since it is x^0 times of the term e^{jx} of $y_c(x)$ and e^{jx} is obtained from the 2 multiple root of the characteristic equation, this is the case 3 on p.227, with $k = 0$ and $r = 2$. Therefore the particular solution contains $x^{k+r}e^{jx} = x^2 e^{jx}$ and all its linearly independent derivatives; $x^2 e^{jx}, xe^{jx}$ and e^{jx}.

Summing up, the triabl function is of the form

$$y_p(x) = Ax^2 + Bx + Cx^2 e^{jx}.$$

($1, xe^{jx}$ and e^{jx} are not included since they already appear in $y_c(x)$.) Its derivatives are

$$\begin{aligned}
y_p'(x) &= 2Ax + B + 2Cxe^{jx} + Cjx^2 e^{jx} \\
y_p''(x) &= 2A + 2Ce^{jx} + 4Cjxe^{jx} - Cx^2 e^{jx} \\
y_p'''(x) &= 6Cje^{jx} - 6Cxe^{jx} - Cjx^2 e^{jx} \\
y_p^{(4)}(x) &= -12Ce^{jx} - 8Cjxe^{jx} + Cx^2 e^{jx} \\
y_p^{(5)}(x) &= -20Cje^{jx} + 10Cxe^{jx} + Cjx^2 e^{jx}
\end{aligned}$$

Substituting these in the differential equation, we get

$$\begin{aligned}
y_p^{(5)} + 2y_p''' + y_p' &= \left(-20Cje^{jx} + 10Cxe^{jx} + Cjx^2e^{jx}\right) \\
&\quad + 2\left(6Cje^{jx} - 6Cxe^{jx} - Cjx^2e^{jx}\right) \\
&\quad + \left(2Ax + B + 2Cxe^{jx} + Cjx^2e^{jx}\right) \\
&= 2Ax + B - 8Cje^{jx} \\
&= 2x + e^{jx}.
\end{aligned}$$

The results are $A = 1, B = 0$ and $C = -1/8j$, therefore a particular solution is

$$y_p(x) = x^2 - \frac{1}{8j}x^2e^{jx} = x^2 + \frac{j}{8}x^2(\cos x + j\sin x) = x^2 + \frac{1}{8}x^2(j\cos x - \sin x).$$

Therefore, the general solution is

$$\begin{aligned}
y =& y_c(x) + \mathrm{Re}\,(y_p(x)) + \mathrm{Im}\,(y_p(x)) \\
=& c_0 + c_1e^{j} + c_2xe^{j} + c_3e^{-j} + c_4xe^{-j} + x^2 + \frac{x^2}{8}(\cos x - \sin x) \\
=& c_0 + (c_1 + c_3)\cos x + j(c_1 - c_3)\sin x + (c_2 + c_4)x\cos x + j(c_2 - c_4)x\sin x \\
& + x^2 + \frac{x^2}{8}(\cos x - \sin x) \\
=& C_0 + C_1\cos x + C_2\sin x + C_3x\cos x + C_4x\sin x + x^2 + \frac{x^2}{8}(\cos x - \sin x)
\end{aligned}$$

where

$$\begin{aligned}
C_0 &:= c_0 \\
C_1 &:= (c_1 + c_3) \\
C_2 &:= j(c_1 - c_3) \\
C_3 &:= (c_2 + c_4) \\
C_4 &:= j(c_2 - c_4).
\end{aligned}$$

- Exercise 22-12 (p.240)

  The characteristic equation of the homogeneous differential equation

$$y'' + 2y' + y = 0$$

is

$$m^2 + 2m + 1 = 0,$$

of which roots are -1 of multiple 2. Therefore the complementary function is

$$y_c(x) = c_1xe^{-x} + c_2e^{-x}.$$

Since the number of linearly independent derivatives of e^{-x}/x is infinite, we need to use variation of parameters method.

Let the trial function be of the form

$$y_p(x) = u_1(x)xe^{-x} + u_2(x)e^{-x}.$$

By (22.27) with $y_1(x) = xe^{-x}, y_2(x) = e^{-x}, Q(x) = e^{-x}/x$ and $a_2 = 1$, we get

$$\begin{aligned} xe^{-x}u_1' + e^{-x}u_2' &= 0 \\ (e^{-x} - xe^{-x})u_1'(x) - e^{-x}u_2'(x) &= e^{-x}/x. \end{aligned}$$

Solving the equations, we get

$$\begin{aligned} u_1'(x) &= 1/x \\ u_2'(x) &= -1. \end{aligned}$$

Therefore, the particular solution is

$$y_p(x) = xe^{-x}\int (1/x)dx + e^{-x}\int dx = xe^{-x}\log x + xe^{-x}$$

and the general solution is

$$y(x) = y_c(x) + y_p(x) = c_1xe^{-x} + c_2e^{-x} + xe^{-x}\log x = e^{-x}(c_1x + c_2 + x\log x).$$

(The term xe^{-x} is not included since it already appears in $y_c(x)$.)

- Exercise 23-20 (p.247)

$$x^2y'' + xy' = 0, \quad x \neq 0.$$

Let $x = e^u$. Due to the hints in the problem 23.18,

$$\begin{aligned} x\frac{dy}{dx} &= \frac{dy}{du} \\ x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{du^2} - \frac{dy}{du}, \end{aligned}$$

therefore the differential equation becomse

$$\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + \frac{dy}{du} = \frac{d^2y}{du^2} = 0.$$

The solution is

$$y(u) = c_1u + c_2$$

therefore

$$y(x) = y(\log x) = c_1\log x + c_2.$$

- Exercise 24-22 (p.267) The differential equation

$$y'' + 3y' + 2y = \sin x$$

can be expressed as

$$(D^2 + 3D + 2)y = (D+2)(D+1)y = \sin x.$$

Let $u = (D+1)y$, then

$$(D+2)u = \sin x.$$

By Lesson 11B, an integraging factor is

$$e^{\int 2dx} = e^{2x}$$

and the solution is

$$e^{2x}y = \int e^{2x} \sin x dx + c = -\frac{1}{5}e^{2x}(\cos x - 2\sin x) + c_1$$

$$\to y = -\frac{1}{5}(\cos x - 2\sin x) + c_1 e^{-2x}.$$

Now we get the differential equation

$$(D+1)y = -\frac{1}{5}(\cos x - 2\sin x) + c_1 e^{-2x}.$$

Again, by Lesson 11B, an integrating factor is

$$e^{\int dx} = e^x$$

and the solution is

$$e^x y = \int e^x \left(-\frac{1}{5}(\cos x - 2\sin x) + c_1 e^{-2x}\right) dx + c_2$$

$$= -\frac{e^x}{10}(3\cos x - \sin x) - c_1 e^{-x} + c_2$$

$$\to y = \frac{1}{10}(\sin x - 3\cos x) - c_1 e^{-2x} + c_2 e^{-x}.$$

- Exercise 25-29 (p.282)

$$(D^3 - 11D^2 + 39D - 45)y = (D-3)^2(D-5)y = e^{3x}.$$

Due to (25.6) with $a = 3, b = 1, r = 2$ and $F(D) = (D-5)$,

$$y_p(x) = \frac{1}{(D-3)^2(D-5)}(e^{3x}) = \frac{x^2 e^{3x}}{2!(3-5)} = -\frac{x^2 e^{3x}}{4}.$$

- Exercise 26-15 (p.292)

  This is equivalent to solving

$$y^{(5)} + 2y''' + y' = 2x + e^{jx}.$$

1. Solving

$$P(D)y = 2x.$$

   By means of partial fraction expansion,

$$\begin{aligned} y_{1p}(x) &:= \frac{1}{P(D)}(2x) \\ &= \frac{1}{D(D^4+2D^2+1)}(2x) \\ &= \frac{1}{D}\left(1 + \frac{-D^4-2D^2}{D^4+2D^2+1}\right)(2x) \\ &= \frac{1}{D}\left(1 - 2D^2 + \frac{2D^6+3D^4}{D^4+2D^2+1}\right)(2x) \\ &= \frac{1}{D}(2x) \\ &= x^2. \end{aligned}$$

2. Solving

$$P(D)y = e^{jx}.$$

   Since

$$\frac{1}{P(D)} = \frac{1}{D(D^2+1)^2} = \frac{1}{D(D+j)^2(D-j)^2},$$

   by (25.6) on p.278 with $a = j, b = 1, r = 2$ and $F(D) = D(D+j)^2$,

$$y_{2p}(x) := \frac{1}{P(D)}(e^{jx}) = \frac{x^2 e^{jx}}{2j(j+j)^2} = \frac{x^2 e^{jx}}{-8j} = \frac{x^2}{8}(j\cos x - \sin x).$$

Therefore, a particular solution is

$$y_p(x) := \mathrm{Re}\,(y_{1p}(x) + y_{2p}(x)) + \mathrm{Im}\,(y_{1p}(x) + y_{2p}(x)) = x^2 + \frac{x^2}{8}(\cos x - \sin x).$$

- Exercise 27-17 (p.311)

$$3y''' + 5y'' + y' - y = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -1.$$

  Comparing the differential equation with (27.2) on p.296, we get

$$n = 3, a_3 = 3, a_2 = 5, a_1 = 1, a_0 = -1, f(x) = 0.$$

Applying Laplace transform on both sides, due to (27.41) on p.299,

$$(a_3s^3 + a_2s^2 + a_1s + a_0)\mathcal{L}[y] - (a_3s^2 + a_2s + a_1)y(0) - (a_3s + a_2)y'(0) - a_3y''(0)$$
$$= (3s^3 + 5s^2 + s - 1)\mathcal{L}[y] - (3s + 5) + 3 = 0$$

$$\to \mathcal{L}[y] = \frac{3s+2}{3s^3 + 5s^2 + s - 1} = \frac{3s+2}{(s+1)^2(3s-1)}.$$

Let

$$\begin{aligned}\mathcal{L}[y] &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{3s-1} \\ &= \frac{A(3s-1)(s+1) + B(3s-1) + C(s+1)^2}{(s+1)^2(3s-1)} \\ &= \frac{(3A+C)s^2 + (2A+3B+2C)s + (-A-B+C)}{(s+1)^2(3s-1)}.\end{aligned}$$

Solving the system of equations

$$\begin{aligned}3A + C &= 0 \\ 2A + 3B + 2C &= 3 \\ -A - B + C &= 2,\end{aligned}$$

we get

$$A = -\frac{9}{16}, B = \frac{1}{4}, C = \frac{27}{16}.$$

Therefore,

$$\begin{aligned}y &= \mathcal{L}^{-1}\left\{\frac{1}{16}\left(-\frac{9}{s+1} + \frac{4}{(s+1)^2} + \frac{27}{3s-1}\right)\right\} \\ &= -\frac{9}{16}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{9}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-1/3}\right\} \\ &= -\frac{9}{16}e^{-x} + \frac{x}{4}e^{-x} + \frac{9}{16}e^{x/3}.\end{aligned}$$

- Exercise 28AB-17 (p.322)

  (a) Take second derivative of (28.25), we get

  $$x''(t) = \frac{d^2x}{dt^2}(t) = -c\omega_0^2 \sin(\omega_0 t + \delta).$$

  Since $|\sin(\omega_0 t+\delta)| \leq 1$, $|x''(t)|$ has its maximum value when $|\sin(\omega_0 t+\delta)| = 1$. Let $|\sin(\omega_0 t_0 + \delta)| = 1$. Then

  $$|x(t_0)| = |c\sin(\omega_0 t_0 + \delta)| = |c|.$$

(b) Since

$$x'(t) = \frac{dx}{dt} = c\omega_0 \cos(\omega_0 t + \delta),$$

the maximum valocity is $c\omega_0$. The maximum acceleration is given above as $c\omega_0^2$.

(c) Due to (28.35) on p.318, the period is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi c\omega_0}{c\omega_0^2} = \frac{2\pi v_m}{a_m}.$$

The amplitude is

$$c = \frac{(c\omega_0)^2}{c\omega_0^2} = \frac{v_m^2}{a_m}.$$

- Exercise 28C-24 (p.331)

(a) Accordint to (28.77) on p.328,

$$\theta(t) = c\cos\left(\sqrt{g/l}t + \delta\right).$$

Since the pendulum starts at the equilibrium position,

$$\theta(0) = c\cos(\delta) = 0$$

therefore

$$\delta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Assuming $c > 0$ and the angle increases when the pendulum starts moving,

$$\theta(t) = c\sin\left(\sqrt{g/l}t + \frac{3\pi}{2}\right) = c\sin\left(\sqrt{g/l}t\right).$$

Moreover, since the velocity

$$\theta'(t) = c\sqrt{g/l}\cos\left(\sqrt{g/l}t\right)$$

is ω_0 at $t = 0$,

$$\theta'(0) = c\sqrt{g/l} = \omega_0 \quad \rightarrow \quad c = \omega_0\sqrt{l/g}$$

therefore

$$\theta(t) = \omega_0\sqrt{l/g}\sin\left(\sqrt{g/l}t\right).$$

(b) Since $-1 \leq \sin(\sqrt{g/l}t) \leq 1$, The pendulum reaches its maximum displacement when $\sqrt{g/l} = \pi/2$. Therefore,

$$\max_t \theta(t) = \omega_0\sqrt{l/g}$$

at $t = \pi\sqrt{l/g}/2$.

- Exercise 28D-15 (p.344)

  Due to the Hooke's law (28.62) on p.324,

$$k = mg/l = 16/8 = 2 \quad \text{(lb/ft)}.$$

  Therefore due to the derivation on p.325, we get the differential equation

$$m\frac{d^y}{dx^2} + ky = f(t) \to \frac{1}{2}\frac{d^y}{dx^2} + 2y = f(t),$$

  since $mg = 16$ and $g = 32\text{ft/sec}^2$. (see p.140)

  1. For $0 \le t \le 1$, the differential equation is

$$y'' + 4y = 2e^t.$$

     The complementary function is

$$y_c(t) = c_1 \sin 2t + c_2 \cos 2t.$$

     This is the case 1 on p.222, therefore the trial function is

$$y_p(t) = c_3 e^t$$

     and we get

$$y_p'' + 4y_p = 5c_3e^t = 2e^t \quad \to c_3 = 2/5.$$

     Therefore the general solution is

$$y(t) = c_1 \sin 2t + c_2 \cos 2t + \frac{2}{5}e^t$$

     and

$$y'(t) = 2c_1 \cos 2t - 2c_2 \sin 2t + \frac{2}{5}e^t.$$

     Since $y(0) = 0$ and $y'(0) = 0$ (the spring is in rest at $t = 0$),

$$y(0) = c_2 + \frac{2}{5} = 0$$
$$y'(0) = 2c_1 + \frac{2}{5} = 0$$

     hence

$$c_1 = -\frac{1}{5} \quad \text{and} \quad c_2 = -\frac{2}{5}$$

     and the general solution is

$$y(t) = \frac{1}{5}(-\sin 2t - 2\cos 2t + 2e^t).$$

2. For $t > 1$:
According to the general solution above, we get the initial conditions

$$y(1) = \frac{1}{5}(-\sin 2 - 2\cos 2 + 2e)$$
$$y'(1) = \frac{1}{5}(-2\cos 2 + 4\sin 2 + 2e).$$

For the differential equation

$$y'' + 4y = 0,$$

which is homogeneous, we have the general solution

$$y(t) = c_4 \sin 2t + c_5 \cos 2t$$

of which derivative is

$$y'(t) = 2c_4 \cos 2t - 2c_5 \sin 2t.$$

Applying the initial condition, we get

$$y(1) = c_4 \sin 2 + c_5 \cos 2 = \frac{1}{5}(-\sin 2 - 2\cos 2 + 2e)$$
$$y'(1) = 2c_4 \cos 2 - 2c_5 \sin 2 = \frac{1}{5}(-2\cos 2 + 4\sin 2 + 2e)$$

of which solution is

$$c_4 = \frac{1}{5}(-1 + 2e\sin 2 + e\cos 2)$$
$$c_5 = \frac{1}{5}(-2 + 2e\cos 2 - e\sin 2)$$

therefore the general solution is

$$y(t) = \frac{1}{5}\left((-1 + 2e\sin 2 + e\cos 2)\sin 2t + (-2 + 2e\cos 2 - e\sin 2)\cos 2t\right).$$

- Exercise 29A-27(a) (p.358)
The characteristic equation of (29.381) is (with variable n)

$$n^2 + (r/m)n + (g/l) = 0$$

of which discriminant is

$$(r/m)^2 - 4(g/l).$$

According to the discussion in Lesson 29A, the system is

1. overdamped if the discriminant is positive:

$$(r/m)^2 - 4(g/l) > 0 \quad \to r^2/(4m^2) > g/l,$$

2. critically damped if the discriminant is zero:

$$r^2/(4m^2) = g/l$$

and

3. underdamped if the discriminant is negative:

$$r^2/(4m^2) < g/l.$$

- Exercise 29B-15 (p.367)

1. For $0 \leq t \leq 1$, the differential equation is

$$y'' + 2y' + 2y = e^{-t}$$

whose complementary function is

$$\begin{aligned} y_c(t) &= \tilde{c}_1 e^{(-1+j)t} + \tilde{c}_2 e^{(-1-j)t} \\ &= e^{-t}(\tilde{c}_1(\cos t + j\sin t) + \tilde{c}_2(\cos t - j\sin t)) \\ &= e^{-t}(c_1 \sin t + c_2 \cos t), \quad (c_1 = \tilde{c}_1 + \tilde{c}_2, c_2 = j(\tilde{c}_1 - \tilde{c}_2)). \end{aligned}$$

Therefore, the trial function is of the form

$$y_p(t) = c_3 e^{-t}$$

hence

$$\begin{aligned} y_p'(t) &= -c_3 e^{-t} \\ y_p''(t) &= c_3 e^{-t}. \end{aligned}$$

Substituting in the differential equation, we get

$$c_3(1 - 2 + 2)e^{-t} = c_3 e^{-t} = e^{-t} \quad \rightarrow y_p(t) = e^{-t}.$$

Therefore, the general solution is

$$y(t) = e^{-t}(1 + c_1 \sin t + c_2 \cos t)$$

hence

$$y'(t) = e^{-t}(-1 + c_1 \cos t - c_2 \sin t - c_1 \sin t - c_2 \cos t).$$

Applying the initial condition $y(0) = y'(0) = 0$, we get

$$\begin{aligned} y(0) &= 1 + c_2 = 0 \\ y'(0) &= -1 + c_1 - c_2 = 0 \end{aligned}$$

of which solution is $c_1 = 0$ and $c_2 = -1$ therefore

$$y(t) = e^{-t}(1 - \cos t).$$

2. According to the general solution above, the initial conditions are

$$y(1) = e^{-1}(1 - \cos 1)$$
$$y'(1) = e^{-1}(-1 + \sin 1 + \cos 1).$$

The general solution of the differential equation

$$y'' + 2y' + 2 = 0,$$

which is homogeneous, is

$$y(t) = e^{-t}(c_3 \sin t + c_4 \cos t)$$

hence

$$y'(t) = e^{-t}(c_3 \cos t - c_4 \sin t - c_3 \sin t - c_4 \cos t).$$

Applying the initial conditions, we get

$$y(1) = e^{-1}(c_3 \sin 1 + c_4 \cos 1) = e^{-1}(1 - \cos 1)$$
$$y'(1) = e^{-1}(c_3 \cos 1 - c_4 \sin 1 - c_3 \sin 1 - c_4 \cos 1) = e^{-1}(-1 + \sin 1 + \cos 1)$$

of which solution is $c_3 = \sin 1$ and $c_4 = \cos 1 - 1$, therefore

$$y(t) = e^{-t} \left(\sin 1 \sin t + (\cos 1 - 1) \cos t\right).$$