Solution of Homework #2

Mathematical Models for Engineering Problems and Differential Equations
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- Excercise 15A, #10. (p.125)

  Let

  - t: time (in minutes) and
  - $x(t)$: the amount of the chemical in the tank A (in galon).

  The initial condition says $x(0) = 300$. Then

  - $x(t)/5000$: the concentration of the liquid in the tank A at time t and
  - $(300 - x(t))/5000$: the concentration of the liquid in the tank A at time t.

  Therefore, due to the circulation, tank A loses $100 * x(t)/5000$ galons and obtains $100 * (300 - x(t))/5000$ galons of the chemical, hence we get the differentil equation:

$$\frac{dx(t)}{dt} = 100\left(-\frac{x(t)}{5000} + \frac{300 - x(t)}{5000}\right) = \frac{1}{25}(150 - x(t))$$

  which can be converted to the separable form

$$\frac{dx(t)}{150 - x(t)} = \frac{dt}{25}.$$

  By integrating both sides separately, we obtain

$$\begin{aligned} & -\log(150 - x(t)) = \frac{t}{25} + c_0 \\ \rightarrow \quad & 150 - x(t) = e^{-\frac{t}{25} - c_0} \\ \rightarrow \quad & x(t) = 150 - c_1 e^{-\frac{t}{25}}. \quad (c_1 = e^{-c_0}) \end{aligned}$$

  Applying the initial condition $x(0) = 300$,

$$x(0) = 150 - c_1 = 300 \quad \rightarrow \quad c_1 = -150,$$

  we get

$$x(t) = 150(1 + e^{-\frac{t}{25}}).$$

(a) Let the tank A contains 200 galons of the liquid at time t_0, then

$$\begin{aligned} & x(t_0) = 150(1 + e^{-\frac{t_0}{25}}) = 200 \\ \rightarrow \quad & e^{-\frac{t_0}{25}} = \frac{200}{150} - 1 = \frac{1}{3} \\ \rightarrow \quad & -\frac{t_0}{25} = -\log 3 \\ \rightarrow \quad & t_0 = 25 \log 3 \approx 25.4653072 \quad \text{(minutes)}. \end{aligned}$$

(b) Impossible, i.e., it takes infinite amount of time since

$$\lim_{t \to \infty} x(t) = 150.$$

- Excercise 15D, #10. (p.133)

  Let

  - t: time (in days) and
  - $x_0(t)$: population of the bacteria at time t in natural state.

  The conditions says $x_0(0) = N_0$ and $x_0(4 \log 2) = 2N_0$.
  Since the increase rate is proportional to the population, we get

  $$\frac{dx_0(t)}{dt} = kx_0(t)$$

  of which solution is

  $$x_0(t) = c_0 e^{kt}.$$

  By applying the condition $x_0(0) = N_0$, we get

  $$x_0(0) = c_0 = N_0.$$

  By applying the condition $x_0(4 \log 2) = 2N_0$, we get

  $$x_0(4 \log 2) = N_0 e^{k4 \log 2} = 2N_0$$

  therefore

  $$e^{k4 \log 2} = 2 = e^{\log 2} \quad \rightarrow k = \frac{1}{4}$$

  and

  $$x_0(t) = N_0 e^{\frac{t}{4}}.$$

  Now, let $x(t)$ be the population of the bacteria when we extract the bacteria at the uniform rate R per day. Then the differential equation becomes

  $$\frac{dx(t)}{dt} = \frac{x(t)}{4} - R$$

which can be converted to the separable form

$$\frac{dx(t)}{x(t) - 4R} = \frac{dt}{4}.$$

By integrating both sides, we obtain

$$\log(x(t) - 4R) = \frac{t}{4} + c_1$$
$$\rightarrow \quad x(t) = c_2 e^{\frac{t}{4}} + 4R. \quad (c_2 = e^{c_1})$$

By applying the initial condition $x(0) = N_0$, we get

$$x(0) = c_2 + 4R = N_0 \quad \rightarrow c_2 = N_0 - 4R.$$

Therefore

$$x(t) = 4\left(R + \left(\frac{N_0}{4} - R\right) e^{\frac{t}{4}}\right).$$

- When $R < N_0/4$, the coefficient of $e^{t/4}$ is positive, therefore the population increases.
- When $R = N_0/4$, $x(t) = 4R$, therefore the population stays the same.
- When $R > N_0/4$, the coefficient of $e^{t/4}$ is negative, therefore the population decreases.

• Excercise 15E, #9. (p.137)

Let

- t: time (in hours) and
- $x(t)$: the amount of the moisture in the substance at time t.

The initial condition says $x(0) = 10$ lb. Note the followings:

- Since the room is sealed, all the moisture the substance lose is contained in the air, i.e., the moisture content of the air at time t is $x(0) - x(t)$.
- When saturated, the air can hold 0.015 lb/ft^3, therefore can hold $0.015 \times 2000 = 30$ lb of moisture.
- Initially, the relative humidity (the ratio of the moisture in the air compared to the saturated amount) of the air is 30%, therefore there is $0.3 \times 30 = 9$ lb of moisture initially.

Now, since the rate of loosing moisture, $dx(t)/dt$, is proportional to

- $x(t)$: its moisture content and
- $30 - (9 + x(0) - x(t))$: the difference between the moisture content of the saturated air, 30, and the moisture content of the air, $9 + x(0) - x(t)$.

Therefore the differential equation is

$$\frac{dx(t)}{dt} = kx(t)(30 - (9 + x(0) - x(t)) = kx(t)(11 + x(t))$$

which can be converted to

$$\begin{aligned} & \frac{dx}{x(11+x)} = kdt \\ \to \quad & \frac{1}{11}\left(\frac{dx}{x} - \frac{dx}{11+x}\right) = kdt \end{aligned}$$

which is separable. By integrating both sides, we get

$$\begin{aligned} & \frac{1}{11}\left(\log x - \log(x+11)\right) = kt + c_0 \\ \to \quad & \log\frac{x}{x+11} = 11kt + 11c_0 \\ \to \quad & \frac{x}{11+x} = c_1 e^{k_1 t} \quad (k_1 = 11k, c_1 = e^{11c_0}) \\ \to \quad & x(t) = \frac{11c_1 e^{k_1 t}}{1 - c_1 e^{k_1 t}} = \frac{11}{c_2 e^{-k_1 t} - 1} \quad (c_2 = 1/c_1) \end{aligned}$$

By applying the initial condition,

$$x(0) = \frac{11}{c_2 - 1} = 10 \quad \to c_2 = 2.1$$

Since it takes 1 hour to lose 4 lb of moisture,

$$\begin{aligned} & x(1) = \frac{11}{2.1e^{-k_1} - 1} = x(0) - 4 = 6. \\ \to \quad & 2.1 \cdot 6e^{-k_1} = 17 \\ \to \quad & k_1 = \log\frac{2.1 \cdot 6}{17}. \end{aligned}$$

To lose 80% of moisture, $10 \times 0.8 = 8$ lb, it takes t_1 hours, i.e.,

$$\begin{aligned} & x(t_1) = \frac{11}{2.1e^{-t_1 \log\frac{2.1\cdot 6}{17}} - 1} = 10 - 8 = 2 \\ \to \quad & 4.2e^{-t_1 \log\frac{2.1\cdot 6}{17}} = 13 \\ \to \quad & -t_1 \log\frac{2.1\cdot 6}{17} = \log\frac{13}{4.2} \\ \to \quad & t_1 = -\frac{\log\frac{13}{4.2}}{\log\frac{2.1\cdot 6}{17}} \approx 3.77229541 \quad \text{(hours).} \end{aligned}$$

- Excercise 16A, #39. (p.156)

  (a) Let

- D: the distance between the earth and the moon,
- m: the mass of the particle,
- M_e: the mass of the earth and
- $M_m = M_e/81$: the mass of the moon.

Then

- F_e, the gravitational attraction by the earth, is

$$F_e = -G\frac{M_e m}{(9D/10)^2} = -G\frac{M_e m}{81(D/10)^2}$$

  and

- F_m, the gravitational attraction by the moon, is

$$F_m = -G\frac{M_m m}{(D/10)^2} = -G\frac{M_e m/81}{(D/10)^2}$$

Since $F_e = F_m$, The particle is at rest.

(b) Let

- r: the distance of the particle a from the center of the earth,
- $v(r)$: the velocity of the particle when the distance is r,
- $a(r)$: the acceleration of the particle when the distance is r,
- R_e: the radius of the earth and
- R_m: the radius of the moon.

Note that the acceleration is

$$a = \frac{dv}{dt} = \frac{dr}{dt}\frac{dv}{dr} = v\frac{dv}{dr}.$$

The force on this particle is composed of the gravitational attraction from the earth and the moon. Considering the direction of each force, we get the differential equation

$$\begin{aligned}
& F = ma = mv\frac{dv}{dr} = -G\frac{M_e m}{r^2} + G\frac{M_m m}{(D-r)^2} \\
\rightarrow \quad & v\frac{dv}{dr} = -G\left(\frac{M_e}{r^2} - \frac{M_m}{(D-r)^2}\right) \\
\rightarrow \quad & v\frac{dv}{dr} = -\frac{gR_e^2}{r^2} + \frac{g_m R_m^2}{(D-r)^2}. \quad \left(g := G\frac{M_e}{R_e^2}, g_m := G\frac{M_m}{R_m^2}\right)
\end{aligned}$$

The differential equation can be converted to the separable form

$$vdv = \left(-\frac{gR_e^2}{r^2} + \frac{g_m R_m^2}{(D-r)^2}.\right) dr$$

Integrating both sides, we get

$$v^2 = \frac{2gR_e^2}{r} + \frac{2g_m R_m^2}{D-r} + C.$$

By applying the initial condition $v(R_e) = v_0$, we get

$$v_0^2 = \frac{2gR_e^2}{R_e} + \frac{2g_m R_m^2}{D - R_e} + C.$$
$$\rightarrow \quad C = v_0^2 - \frac{2gR_e^2}{R_e} - \frac{2g_m R_m^2}{D - R_e}.$$

Therefore

$$(v(r))^2 = \frac{2gR_e^2}{r} + \frac{2g_m R_m^2}{D - r} + v_0^2 - 2gR_e - \frac{2g_m R_m^2}{D - R_e}.$$

(c) As the hint says, we want the velocity to be zero when the particle reaches the 'neutral' point, $r = 9D/10$. Therefore, with the conditions $R_m^2 = 6R_e^2/81$, $D = 61R_e + R_e/4 = \frac{7^2 5}{4} R_e$ and $g_m = g/6$, we get

$$\begin{aligned}
(v(9D/10))^2 &= \frac{2gR_e^2}{9D/10} + \frac{2g_m R_m^2}{D - (9D/10)} + v_0^2 - 2gR_e - \frac{2g_m R_m^2}{D - R_e} \\
&= \frac{2gR_e^2}{9D/10} + \frac{1}{6}\frac{6}{81}\frac{2gR_e^2}{D/10} + v_0^2 - 2gR_e - \frac{1}{6}\frac{6}{81}\frac{2gR_e^2}{\frac{(7^2 5 - 4)R_e}{4}} \\
&= \left(\frac{2 \cdot 10 \cdot 4}{9 \cdot 7^2 5} + \frac{2 \cdot 10 \cdot 4}{81 \cdot 7^2 5} - 2 - \frac{2 \cdot 4}{81 \cdot 241} \right) gR_e + v_0^2 \\
&\approx -1.96 gR_e + v_0^2 = 0.
\end{aligned}$$

Therefore,

$$v_0 = \sqrt{1.96 gR_e} \approx (0.99)\sqrt{2gR_e}.$$

Note that $\sqrt{2gR_e}$ is the 'escape velocity' found in the Example 16.36, #4. (p.146)

- Excercise 17B, #6. (p.176)

In the figure, the slope of the tractrix is (keeping in mind that $y \le l$)

$$\frac{dy}{dx} = \tan\theta = \frac{y}{-\sqrt{l^2 - y^2}}.$$

(Note that, since $\theta > \pi/2$ in the figure, $\tan\theta < 0$ therefore we should take $-\sqrt{l^2 - y^2}$ not $\sqrt{l^2 - y^2}$.)

Let $u^2 = l^2 - y^2$ hence

$$2udu = -2ydy \quad \rightarrow dy = -\frac{udu}{y}.$$

The differential equation becomes a separable form

$$
\begin{aligned}
& \frac{udu}{y} = \frac{ydx}{u} \\
\rightarrow \quad & \frac{u^2du}{l^2-u^2} = dx \\
\rightarrow \quad & \left(\frac{u}{l-u} - \frac{u}{l+u}\right) du = 2dx \\
\rightarrow \quad & \left(\frac{l}{l-u} + \frac{l}{l+u} - 2\right) du = 2dx.
\end{aligned}
$$

By integrating both sides, we obtain

$$
\begin{aligned}
& -l\log|l-u| + l\log|l+u| - 2u = 2x + C. \\
\rightarrow \quad & l\log\frac{l+u}{l-u} - 2u = 2x + C \\
\rightarrow \quad & x = \frac{l}{2}\log\frac{(l+u)^2}{l^2-u^2} - u - C/2 \\
\rightarrow \quad & x = l\log\frac{l+u}{\sqrt{l^2-u^2}} - u - C/2 \\
\rightarrow \quad & x = l\log\frac{l+\sqrt{l^2-y^2}}{y} - \sqrt{l^2-y^2} - C/2.
\end{aligned}
$$

Since the boat was at $(0, l)$ initially,

$$0 = -C/2 \quad \rightarrow C = 0.$$

Therefore the tractrix is

$$x = l\log\frac{l+\sqrt{l^2-y^2}}{y} - \sqrt{l^2-y^2}.$$