Solution of Homework #1

Mathematical Models for Engineering Problems and Differential Equations
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15 $(2y - xy\log x)dx - 2x\log x dy = 0$

The DE implies the domain $x > 0$ by "$\log x$." By dividing wth $xy\log x$ for $x \neq 1$ and $y \neq 0$, we get a separable one

$$\left(\frac{2}{x\log x} - 1\right)dx - \frac{2}{y}dy = 0. \tag{1}$$

Note that $\displaystyle\int \frac{1}{x\log x} = \log|\log x| + C$ since

- for $x > 1$,

$$\frac{d(\log|\log x|)}{dx} = \frac{d(\log(\log x))}{dx} = \frac{1}{\log x}\frac{d(\log x)}{dx} = \frac{1}{x\log x}$$

and

- for $0 < x < 1$,

$$\frac{d(\log|\log x|)}{dx} = \frac{d(\log(-\log x))}{dx} = \frac{1}{-\log x}\frac{d(-\log x)}{dx} = \frac{1}{x\log x}.$$

Therefore, we get the 1-parameter family of solutions

$$2\log|\log x| - x - 2\log|y| = C. \tag{2}$$

We need to check if there is any particular solution that cannot be expressed by (2). When we convert the original DE to (1), we excluded $y = 0$. Since this is a particular solution and cannot be expressed by (2). (Note that (2) requires $y \neq 0$.)

21 $xy' - y^2 + 1 = 0$ We can convert the original DE to (for $x \neq 0$ and $|y| \neq 1$)

$$\frac{1}{x}dx - \frac{1}{y^2 - 1}dy = 0$$

which is separable. Since

$$\frac{1}{y^2-1} = \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right),$$

we get the 1-parameter family of solutions

$$\log|x| - \frac{1}{2}\log\left|\frac{y+1}{y-1}\right| = C.$$

This can be converted to

$$\log\left|x^2\frac{y-1}{y+1}\right| = C$$

therefore

$$y = \frac{cx^2+1}{cx^2-1} \tag{3}$$

where $c = e^C > 0$. $|y| = 1$ are particular solutions. $y = 1$ can be expressed by (3) (with $c = 0$), but not $y = -1$.

27 $xy' + ay + bx^n = 0, \quad x > 0$

The original DE can be converted to a linear one

$$y' + \frac{a}{x}y = -bx^{n-1}$$

where $P(x) = a/x$ and $Q(x) = -bx^{n-1}$. Since

- $\displaystyle\int P(x)dx = \int \frac{a}{x}dx = a\log|x|$ and
- $\displaystyle\int e^{\int P(x)dx}Q(x)dx = \int -bx^{a+n-1}dx = \begin{cases} \frac{-b}{n+a}x^{n+a} & (n+a \neq 0) \\ -b\log|x| & (n+a = 0), \end{cases}$

the 1-parameter family of solutions is

$$\begin{cases} y = x^{-a}\left(\frac{-b}{n+a}x^{n+a}\right) + cx^{-a} = \frac{-b}{n+a}x^n + cx^{-a} & (n+a \neq 0) \\ y = -bx^{-a}\log x + cx^{-a} & (n+a = 0) \end{cases}$$

33 $(x^2y - 1)y' + xy^2 - 1 = 0$

Since

$$d(x^2y^2) = 2xy^2dx + 2x^2ydy,$$

The original DE can be converted to the exact one

$$d(x^2y^2) - 2dx - 2dy = 0.$$

Therefore the 1-parameter family of solutions is

$$x^2y^2 - 2x - 2y = c.$$

39 $(x^2 - y)y' + x = 0$

If we regard y as the independent variable, the original DE can be converted to

$$(x^2 - y) + x\frac{dx}{dy} = 0.$$

By dividing by x, we get

$$\frac{dx}{dy} + x = yx^{-1}$$

which is a Bernoulli equation with $P(y) = 1$, $Q(y) = y$ and $n = -1$. By multiplying $(1-n)x^{-n} = 2x$ to both sides, we get

$$2x\frac{dx}{dy} + 2x^2 = \frac{d(x^2)}{dy} + 2x^2 = 2y.$$

With $u = x^2$, this becomes a linear DE

$$u' + 2u = 2y$$

where $\widetilde{P}(y) = 2$ and $\widetilde{Q}(y) = 2$ and of which the 1-parameter family of solutions is (by textbook (11.191))

$$\begin{aligned} u(y) &= e^{-\int \widetilde{P}(y)dy} \int e^{\int \widetilde{P}(y)dy} \widetilde{Q}(y)dy + ce^{-\int \widetilde{P}(y)dy} \\ &= e^{-2y} \int 2e^{2y} y dy + ce^{-2y} \\ &= e^{-2y} \frac{1}{2} \left(2ye^{2y} - e^{2y}\right) + ce^{-2y} \\ &= \frac{1}{2}(2y - 1) + ce^{-2y}. \end{aligned}$$

Therefore, the 1-parameter family of solutions is

$$2x^2 = 2y - 1 + c_0 e^{-2y}$$

where $c_0 = 2c$.

45 $(x^2 + y^2)y' + 2x(2x + y) = 0$

The original DE can be converted to

$$(x^2 y' + 2xy) + y^2 y' + 4x^2 = 0.$$

Since $d(x^2 y) = 2xydx + x^2 dy$, the DE can again converted to

$$d(x^2 y) + y^2 dy + 4x^2 dx = 0$$

which is exact. The 1-parameter family of solutions is

$$3x^2 y + y^3 + 4x^3 = c.$$