

Computer Graphics

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Appendix B: Spaces

B.1: Scalars

Scalar Field

- ▶ *Closed* under the two fundamental operations:
 $\forall \alpha, \beta \in S, \alpha + \beta \in S$ and $\alpha \cdot \beta \in S$. (S is the set of scalars)
- ▶ associative, commutative, and distributive
 - ▶ $\alpha + \beta = \beta + \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha$
 - ▶ $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ and $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
 - ▶ $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$
- ▶ additive identity (0) and multiplicative identity (1)
 - ▶ $\alpha + 0 = 0 + \alpha = \alpha$
 - ▶ $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$
- ▶ additive and multiplicative inverses
 - ▶ $\alpha + (-\alpha) = 0$
 - ▶ $\alpha \cdot \alpha^{-1} = 1$
- ▶ Examples: real numbers, complex numbers, rational functions, etc.

B.2: Vector Spaces

Vector Spaces (V)

- ▶ Scalars \cup vectors
- ▶ Vector operations
 - ▶ Vector-vector addition
 - ▶ $\forall u, v \in V, u + v \in V$
 - ▶ Zero vector ($\mathbf{0}$): $\forall u \in V, u + \mathbf{0} = u$
 - ▶ Additive inverse: $\forall u \in V, u + (-u) = \mathbf{0}$
 - ▶ Scalar-vector multiplication
 - ▶ $\forall u \in V$ and $\forall \alpha \in S, \alpha u \in V$
 - ▶ Distributive. $\forall u, v \in V$ and $\forall \alpha, \beta \in S,$
 $\alpha(u + v) = \alpha u + \alpha v$ and $(\alpha + \beta)u = \alpha u + \beta u$
- ▶ Examples: geometric vectors, \mathbb{R}^n (n -tuples of real numbers), etc.

Vector Spaces (cont'd)

- ▶ Linear combination: $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in V$,
($\alpha_1, \cdots, \alpha_n \in S$, $u_1, \cdots, u_n \in V$)
- ▶ u_1, \cdots, u_n are *linearly independent* if $\alpha_1 u_1 + \cdots + \alpha_n u_n = \mathbf{0}$ only when $\alpha_1 = \cdots = \alpha_n = 0$. In other words, for any vector $v \in V$, the linear combination of u_1, \cdots, u_n such that $v = \alpha_1 u_1 + \cdots + \alpha_n u_n$ is *unique*.
- ▶ The maximum number of linearly independent vectors we can find in a space is the *dimension* of the space.
- ▶ For an n -dimensional space, *any* n linearly independent vectors form a *basis*. (examples?)
- ▶ If $\{v_j\}_{j=1}^n$ is a basis of V , any vector $v \in V$ can be *uniquely* expressed as $v = \sum_{j=1}^n \beta_j v_j$. The scalars $\{\beta_j\}_{j=1}^n$ give the *representation* of v with respect to the basis $\{v_j\}_{j=1}^n$.
- ▶ For a fixed basis, we can work only with representations. (*coordinate system*)

B.3: Affine Spaces

Affine Spaces

- ▶ We cannot represent points in vector space!
- ▶ Scalars \cup vectors \cup *points*
- ▶ Point-point subtraction: for the points P and Q , $v = P - Q$ is a vector. Therefore, for any vector v and point P , $Q = v + P$ is a point.
- ▶ Frame (coordinate system \cup *reference point (origin)*)
- ▶ For a frame with basis $\{v_j\}_{j=1}^n$ and origin P_0 ,
 - ▶ any vector v can be uniquely written as $v = \sum_{j=1}^n \alpha_j v_j$, hence $\{\alpha_j\}$ is the representation of v , and
 - ▶ any point P can be uniquely written as $P = P_0 + \sum_{j=1}^n \beta_j v_j$, hence $\{\beta_j\}$ is the representation of P .
- ▶ No scalar-point multiplication! (except special cases)

B.4: Euclidean Spaces

Euclidean Spaces

- ▶ Vector space with *inner (dot) product*
- ▶ Inner (dot) product. For $\alpha, \beta \in \mathcal{S}$, $u, v, w \in V$,
 - ▶ $u \cdot v = v \cdot u$
 - ▶ $(\alpha u + \beta v) \cdot w = \alpha u \cdot w + \beta v \cdot w$
 - ▶ $v \cdot v > 0$ if $v \neq 0$
 - ▶ $\mathbf{0} \cdot \mathbf{0} = 0$
- ▶ u and v are *orthogonal* if $u \cdot v = 0$.
- ▶ The *magnitude (length)* of a vector v is measured as $|v| = \sqrt{v \cdot v}$.
- ▶ *Distance* of two points?